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Master Equation Description of External Poisson White Noise in Finite Systems

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We consider finite systems with random control parameters. A theory for a unified description of internal fluctuations and external noise is presented. Internal fluctuations are modeled by a one-step Markovian master equation. External noise is introduced by random parameters in the master equation. It is modeled by a Poisson white noise. The unified description of fluctuations features a Markovian master equation with nonvanishing transition probabilities for all steps in the state space. Alternative formulations are given in terms of the generating function, Poisson representation and the equations for the factorial moments. An expansion around the thermodynamic limit is considered. The theory permits the calculation of finite-size effects. It predicts the existence of a coupling of the two types of fluctuations leading to "crossed-fluctuation" contributions. Two examples are considered: (i) a Poisson counting process with fluctuating parameter, (ii) a creation and annihilation process with source terms and fluctuations in each of the creation, annihilation, and source parameters. In the second example a complete analysis is given for the stationary distribution and associated moments for a finite system and also in the thermodynamic limit. The different role of the fluctuations of the three parameters is discussed. Explicit "crossed-fluctuations" contributions are found. The effect of the system size on the type of transitions induced by external noise in the thermodynamic limit is discussed.

KEY WORDS: Fluctuations; external noise; master equation; Poisson white noise.

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1. INTRODUCTION

In the last decade several authors have studied nonequilibrium system whose control parameters vary randomly in time. This general physical situation is nowadays known as the external noise problem. It has been predicted that external noise can modify in nontrivial ways the behavior of the system. The effect is particularly important in the vicinity of non-equilibrium transitions. The theoretical description of such systems as well as several experimental studies have been reviewed in a recent monograph by Horsthemke and Lefever.[1]

The standard theoretical description of the effects of external noise features a set of stochastic differential equations of first order in time for the relevant macroscopic variables of the system. Usually only one variable is considered. Such stochastic differential equations are obtained from a deterministic description of the system in which the precisely defined value of a control parameter is replaced by a stochastic process. This process models the external noise. The starting deterministic description emerges as a limit of a more refined description of the system. At this more detailed level the intrinsic fluctuations of the system are taken into account. The deterministic description is obtained in the thermodynamic limit in which the internal fluctuations disappear. Therefore, the underlying assumption of the standard external noise description is that internal fluctuations can be neglected in comparison with the ones originated by external noise. This is a very reasonable assumption for an ordinary macroscopic system. But for a finite system there will be a contribution from the internal fluctuations which can be important in thermodynamically small systems as microdevices. Moreover, it is conceivable that there exists some kind of coupling of internal and external fluctuations which cannot be studied in the thermodynamic limit. Lastly, from a first-principles point of view there exists the desire of having a description in which internal and external fluctuations are simultaneously and consistently considered. In this paper we present such a description. Within it, finite-size effects in a system driven by external noise can be calculated and coupling effects of internal and external fluctuations are found.

We describe the internal fluctuations at a mesoscopic level by a discrete Markovian master equation.[2] For simplicity we only consider one-step processes. In the thermodynamic limit the master equation reduces to a deterministic description.[3–5] The control parameters appear as constant coefficients in the transition probabilities of the master equation. External noise is introduced at this level by replacing such constant coefficients by random processes. This leads to stochastic transition probabilities. The average of the master equation with stochastic transition probabilities over the realizations of the random processes modeling the external noise results in a new master equation. This final equation gives the desired unified description of fluctuations only in terms of the system variables. A first attempt in this direction was reported by San Miguel and Sancho[6] using Gaussian white noise as a model of external noise. In this way, sensible answers for the statistical properties of the system are found. Nevertheless, the procedure is not completely satisfactory because the stochastic transition probabilities become negative for some realizations of the noise. To avoid this shortcoming a deeper analysis based on more safe grounds is needed. The joint description of fluctuations has also been considered using a dichotomic Markov process for the external noise.[7] This permits to maintain the positivity of the stochastic transition probabilities. The difficulties in this case are that the dichotomic Markov process is not a realistic model of natural external noise and that the relevant physical variable does not follow a Markovian process. An application to nuclear reactor models with external Gaussian white noise and dichotomic Markov noise is given in Ref. 8. In this paper we present a complete analysis of the problem of a joint description of fluctuations modeling the external noise by a Poisson white noise.[9–16] In spite of the technical difficulties of dealing with such a noise, it permits to maintain the positivity of the stochastic transition probabilities and the physical variable follows a Markov process which incorporates internal and external fluctuations.

In Section 2 we review the standard description in the thermodynamic limit for Poisson external white noise. In Section 3 we present the general formulation of the problem. We derive the master equation for the unified description of fluctuations. Starting from a one-step master equation, it is found that if the fluctuating coefficient belongs to the transition probability associated with a one step jump forward (backward) in state space, the averaged master equation includes nonvanishing transition probabilities for all possible jumps forward (backward) in state space. It is difficult to extract relevant statistical information in concrete cases from this master equation. To overcome this difficulty, we develop three other equivalent alternative mathematical formulations of the problem: the equation for the generating function, the Poisson representation of the master equation, and the equations for the factorial moments. In particular, from the equation for the moments we show the existence of "crossed-fluctuation" contributions which couple internal and external fluctuations.[3] They vanish

[3] After this work was submitted for publication, Horsthemke and Lefever [*Phys. Lett.* **106A**:10 (1984)] proposed a model to give a joint description of internal and external fluctuations, based on the use of stochastic differential equations. We note that in such model "crossed-fluctuations" only occur at finite correlation times of the external fluctuations while here they are found for a white external noise. (For a discussion in this context of the passage from a master equation to stochastic differential equation see the second paper in Ref. 8.)

both in the thermodynamic limit and in the absence of external noise. Finally we present an expansion of our general equations around the thermodynamic limit. This gives a systematic method for the calculation of finite size effects. In Section 4 we discuss several examples. We first discuss the Poisson counting process whose parameter undergoes fluctuations given by a Poisson white noise. Our second example is a creation and annihilation process with source. This process has been used to study the internal fluctuations in maser amplification,[17] a chemical model,[18] and a nuclear reactor model.[19] We consider external noise separately in the source parameter (additive noise) and in the annihilation and in the creation parameter (multiplicative noise). We first discuss these three cases in the thermodynamic limit. To our knowledge this study has not been previously reported in the literature for Poisson white noise. In three cases we find a sort of "noise-induced transition"[1] which does not exist for Gaussian white noise. In the multiplicative noise cases the conditions for the existence of stationary state and stationary moments can be understood in terms of realizations of the noise which destabilize the system. The behavior of the system is different for fluctuations of the annihilation parameter and fluctuations of the creation parameter. This difference disappears if external noise is modeled by a Gaussian white noise, because in this latter case the positivity of the parameters cannot be maintained. On the other hand, with Poisson white noise we find a similar behavior of the stationary distribution when fluctuations are considered either in the source parameter or in the creation parameter. For finite systems, and in the three cases, we calculate the stationary distributions which incorporate internal and external fluctuations, and also their associated moments. Exact analytical results are given for fluctuations in the source and annihilation parameters. Numerical results for the probability distribution are shown for the three cases. They are obtained by an iterative calculation outlined in Appendix C. The conditions for the existence of a stationary distribution and stationary moments are not changed by the consideration of a finite volume in any of the three cases. Also the mean value is not modified from its value in the thermodynamic limit. The relative fluctuation around the mean value is in the first case (additive noise) the sum of its value in the thermodynamic limit with the one in the absence of external noise. In the two other cases (multiplicative noise) we find an additional "crossed-fluctuation" contribution. The dependence that we find in the three cases of the mean value and relative fluctuations on the noise parameters and the volume of the system is discussed in the light of our general results of Section 3. The transitions found in the behavior of the stationary distributions, in the thermodynamic limit, are smeared out in a finite system. Important changes can still be seen in a finite but large system. For small system sizes internal fluctuations dominate, no noticeable transition is seen, and the stationary distribution is quite independent of which parameter fluctuates. The similar form of the stationary distribution for fluctuations of the source parameter or fluctuations of the creation parameter is maintained for all values of the system size. Some properties of the white Poisson noise are summarized in Appendix A. Appendix B contains a proof of an important operator relation used in the main text. The consequences of our results in the Gaussian white noise limit of the external noise will be discussed in detail elsewhere.

2. WHITE POISSON NOISE IN THE THERMODYNAMIC LIMIT

In this section we first summarize the master equation description of internal fluctuations. Secondly we discuss the framework used to discuss the effect of white Poisson external noise in the thermodynamic limit of the master equation.

2.1. Master Equation

Intrinsic fluctuations of a finite homogeneous system can be described in many systems[2,7] by a Markovian master equation of the general form

$$\begin{aligned} \frac{\partial P(N,t)}{\partial t} &= \sum_{n=1}^{\infty} \{W(N, N-n)\, P(N-n, t) + W(N, N+n)\, P(N+n, t)\} \\ &\quad - \sum_{n=1}^{\infty} \{W(N+n, N) + W(N-n, N)\}\, P(N, t) \end{aligned} \tag{2.1}$$

where $W(N \pm n, N)$ is the transition probability of n steps, from state N to state $N \pm n$. Here we restric ourselves to one-step processes. Introducing the notation

$$Q(N) \equiv W(N+1, N) \tag{2.2}$$

$$R(N) \equiv W(N-1, N) \tag{2.3}$$

and the operators

$$E^{\pm} = \exp\left\{\pm \frac{\partial}{\partial N}\right\} \tag{2.4}$$

the master equation can be written for one-step processes as

$$\frac{\partial P(N,t)}{\partial t} = [(E^{-}-1)\,Q(N) + (E^{+}-1)\,R(N)]\,P(N,t)$$

$$\equiv \Gamma\left(\frac{\partial}{\partial N}, N\right) P(N,t) \tag{2.5}$$

Two useful representations of the master equation which we will use later are given by the generating function $F(s,t)$[2,20,21] and the Poisson transform $f(a,t)$.[18] They are defined by

$$F(s,t) = \sum_{N=0}^{\infty} s^N P(N,t) \tag{2.6}$$

$$P(N,t) = \int da\, e^{-a} \frac{a^N}{N!} f(a,t) \tag{2.7}$$

The equation satisfied by $F(s,t)$ is[21]

$$\frac{\partial F(s,t)}{\partial t} = \left[(s-1)\,Q\left(s\frac{\partial}{\partial s}\right) + \left(\frac{1}{s}-1\right) R\left(s\frac{\partial}{\partial s}\right)\right] F(s,t) \tag{2.8}$$

In order to write down the equation for $f(a,t)$ we assume without loss of generality that the transition probabilities $Q(N)$ and $R(N)$ can be written as a sum of factorial products $\Omega_m(N)$

$$Q(N) = \sum_{m=0}^{m_0} \delta_m \Omega_m(N) \tag{2.9}$$

$$R(N) = \sum_{l=1}^{l_0} \nu_l \Omega_l(N) \tag{2.10}$$

where

$$\Omega_m(N) = N(N-1)\cdots(N-m+1) \tag{2.11}$$

In this case the equation for $f(a,t)$ is

$$\frac{\partial f(a,t)}{\partial t} = \left[-\sum_{m=0}^{m_0} \hat{Q}^m\left(a, \frac{\partial}{\partial a}\right) + \sum_{l=1}^{l_0} \hat{R}^l\left(a, \frac{\partial}{\partial a}\right)\right] f(a,t) \tag{2.12}$$

where

$$\hat{Q}^m\left(a, \frac{\partial}{\partial a}\right) = \delta_m \frac{\partial}{\partial a}\left(1 - \frac{\partial}{\partial a}\right)^m a^m \tag{2.13}$$

$$\hat{R}^l\left(a, \frac{\partial}{\partial a}\right) = v_l \frac{\partial}{\partial a}\left(1 - \frac{\partial}{\partial a}\right)^{l-1} a^l \tag{2.14}$$

Equations (2.5), (2.8), and (2.12) are equivalent standard starting points in the study of intrinsic fluctuations of a system.

In the thermodynamic limit in which the volume of the system $V \to \infty$, with $N \to \infty$ and $x = N/V$ finite, intrinsic fluctuations disappear and (2.5) reduces to a deterministic description defined by a macroscopic evolution equation for x.[3-5] The transition probabilities are assumed to be extensive quantities

$$Q(N) = Vq(x), \qquad R(N) = Vr(x) \tag{2.15}$$

In terms of $q(x)$ and $r(x)$ the deterministic equation in the thermodynamic limit is

$$\dot{x} = q(x) - r(x) \tag{2.16}$$

In (2.15) $q(x)$ and $r(x)$ may contain contributions proportional to V^{-n} which are neglected in (2.16).

2.2. External Noise in the Thermodynamic Limit

The standard description of external noise effects[1] is made at the level of the deterministic equation (2.16) by replacing constant parameters in (2.16) by random functions of time. We assume here that $q(x) = q_0^0(x) + \alpha q_1(x)$ contains a parameter α and $r(x) = r_0^0(x) + \beta r_1(x)$ a parameter β which become random functions of time according to

$$\alpha = \bar{\alpha} + \xi_Q(t), \qquad \beta = \bar{\beta} + \xi_R(t) \tag{2.17}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are, respectively, the mean values of α and β and $\xi_Q(t)$ and $\xi_R(t)$ are random processes of zero mean value. α and β are assumed here to be positive definite parameters and also intensive quantities. When α and β are given by (2.17), $q(x)$ and $r(x)$ become

$$q(x) = q_0(x) + \xi_Q(t)\, q_1(x) \tag{2.18}$$

$$r(x) = r_0(x) + \xi_R(t)\, r_1(x) \tag{2.19}$$

where

$$q_0(x) = q_0^0(x) + \bar{\alpha} q_1(x) \tag{2.20}$$

$$r_0(x) = r_0^0(x) + \bar{\beta} r_1(x) \tag{2.21}$$

Substituting (2.18) and (2.19) in (2.16) we obtain a stochastic differential equation (SDE) which is the standard starting point in the study of external noise. It is more convenient to deal with the equation for the probability density $P(x, t)$ of the process than with the SDE for $x(t)$.

$P(x, t)$ is given by

$$P(x, t) = \overline{\delta(x(t) - x)} \tag{2.22}$$

where $\overline{(\cdots)}$ indicates the average over the realizations of $\xi_Q(t)$ and $\xi_R(t)$. From (2.16), (2.18), (2.19), and (2.22) we obtain

$$\begin{aligned} \frac{\partial \delta(x(t) - x)}{\partial t} = & -\frac{\partial}{\partial x} [q_0(x) - r_0(x)]\, \delta(x(t) - x) \\ & - \frac{\partial}{\partial x} q_1(x)\, \xi_Q(t)\, \delta(x(t) - x) \\ & + \frac{\partial}{\partial x} r_1(x)\, \xi_R(t)\, \delta(x(t) - x) \end{aligned} \tag{2.23}$$

The final closed equation for $P(x, t)$ is obtained when the average over $\xi_Q(t)$ and $\xi_R(t)$ is explicitly performed. The result depends, of course, in the statistical properties of $\xi_Q(t)$ and $\xi_R(t)$. In this paper we consider that both $\xi_Q(t)$ and $\xi_R(t)$ are independent white generalized Poisson noise (white shot noise).

White Poisson noise has been considered in recent years in Refs. 9–16. It is obtained as a white noise limit of a generalized Poisson process $z(t)$ (see Appendix A). It represents a sequence of delta peaks at random points in time. These random points are given by a Poisson counting process with parameter λ. The average time difference between two peaks is controlled by λ and the amplitude ω of the peaks is distributed according to a probability density $\rho(\omega)$. White Poisson noise $z^w(t)$ can be defined by its characteristic (moment-generating) functional with test function $v(t_1)$ (see Appendix A)

$$\Phi_t^w[v] = \exp\left[\lambda \int_0^t dt_1 (\{e^{i\omega v(t_1)}\}_{\text{av}} - 1)\right] \tag{2.24}$$

where $\{\cdots\}_{\text{av}}$ indicates the average taken with $\rho(\omega)$. The process $z^w(t)$ has mean value $\lambda\bar{\omega}$, with $\bar{\omega}=\{\omega\}_{\text{av}}$. The white Poisson noise $\xi(t)$ that we will consider has zero mean value. It is defined by $\xi(t)=z^w(t)-\lambda\bar{\omega}$. An important feature of $\xi(t)$ is that it is bounded from below $\xi(t)\geqslant -\lambda\bar{\omega}$. This permits that a parameter α fluctuates according to (2.17) keeping a definite sign. This property is lost in the Gaussian white noise limit defined by $\lambda\to\infty$, $\{\omega^n\}_{\text{av}}\to 0$ with $\lambda\bar{\omega}\to\infty$, $\lambda\{\omega^2\}_{\text{av}}=2D$ and $\lambda\{\omega^n\}_{\text{av}}\to 0$ for $n\geqslant 3$. Other properties of the white Poisson noise are summarized in Appendix A. In this Appendix we also derive an important formula for the average of $\xi(t)$ with a functional $\varphi[\xi]$:

$$\overline{\xi(t)\,\varphi[\xi]}=\lambda\sum_{n=2}^{\infty}\frac{\{\omega^n\}_{\text{av}}}{n!}\frac{\delta^{n-1}\varphi}{\delta^{n-1}\xi(t)} \tag{2.25}$$

Making use of this formula it is easy to take the average of equations given for each realization of $\xi(t)$. Consider a general linear equation of the form

$$\frac{\partial A(x,t)}{\partial t}=O_1A(x,t)+\xi(t)\,O_2A(x,t) \tag{2.26}$$

where O_1 and O_2 are operators acting on A which do not depend on $\xi(t)$. Equation (2.26) implies that $A(x,t)$ is a functional of $\xi(t)$ such that

$$\frac{\delta^n A(x,t)}{\delta\xi(t)^n}=O_2^nA(x,t) \tag{2.27}$$

Taking the average of (2.26) we immediately obtain from (2.25) and (2.27) that

$$\frac{\partial\bar{A}(x,t)}{\partial t}=O_1\bar{A}(x,t)+\lambda[\{e^{\omega O_2}\}_{\text{av}}-\bar{\omega}O_2-1]\,\bar{A}(x,t) \tag{2.28}$$

where $\bar{A}(x,t)$ is the average of $A(x,t)$ over the realizations of $\xi(t)$.

With the use of (2.28) for the two independent processes ξ_Q and $\xi_R(t)$ we obtain for the average of (2.23)

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{\partial}{\partial x}[q_0(x)-r_0(x)]\,P(x,t)\\&+\lambda_Q\left[\{e^{-\omega_Q(\partial/\partial x)q_1(x)}\}_{\text{av}}+\bar{\omega}_Q\frac{\partial}{\partial x}q_1(x)-1\right]P(x,t)\\&+\lambda_R\left[\{e^{\omega_R(\partial/\partial x)r_1(x)}\}_{\text{av}}-\bar{\omega}_R\frac{\partial}{\partial x}r_1(x)-1\right]P(x,t)\\\equiv{}& L_0P(x,t)\end{aligned} \tag{2.29}$$

This equation describes, in the thermodynamic limit, the effect of external noise in the parameters α and β. In the following we will consider two particular choices of the distribution $\rho(\omega)$. The first one is the case in which ω takes a single value $\bar{\omega}$: $\rho(\omega)=\delta(\omega-\bar{\omega})$. The equation for $P(x, t)$ is trivially obtained from (2.29) in this case. The second case is an exponential distribution

$$\rho(\omega)=\frac{e^{-\omega/\bar{\omega}}}{\bar{\omega}} \tag{2.30}$$

where ω is restricted to take positive values, in this case (2.29) becomes

$$\begin{aligned}\frac{\partial P(x, t)}{\partial t}= &-\frac{\partial}{\partial x}\left[q_0(x)-r_0(x)-\lambda_Q\bar{\omega}_Q q_1(x)+\lambda_R\bar{\omega}_R r_1(x)\right] P(x, t)\\ &-\lambda_Q\bar{\omega}_Q\frac{\partial}{\partial x} q_1(x)\frac{1}{1+\bar{\omega}_Q(\partial/\partial x)\, q_1(x)} P(x, t)\\ &+\lambda_R\bar{\omega}_R\frac{\partial}{\partial x} r_1(x)\frac{1}{1-\bar{\omega}_R(\partial/\partial x)\, r_1(x)} P(x, t)\end{aligned} \tag{2.31}$$

The stationary solution of (2.31) is easily found in the case in which there is a single fluctuating parameter. For $q_1=0$ we have[16]

$$\begin{aligned}P_{st}(x)\sim &\left[q_0(x)-r_0(x)+\lambda_R\bar{\omega}_R r_1(x)\right]^{-1}\\ &\times\exp\left\{\int^x\frac{q_0(x')-r_0(x')}{\bar{\omega}_R r_1(x')\left[q_0(x')-r_0(x')+\lambda_R\bar{\omega}_R r_1(x')\right]}\,dx'\right\}\end{aligned} \tag{2.32}$$

The equations satisfied by the moments $\overline{x^m}_t$ are easily derived from (2.31):

$$\begin{aligned}\frac{d}{dt}\overline{x^m}_t= &\, m\,\overline{x^{m-1}\left[q_0(x)-r_0(x)\right]}_t+\lambda_Q\sum_{k=2}^{\infty}\bar{\omega}_Q^k\,\overline{\left(q_1(x)\frac{d}{dx}\right)^k x^m}_t\\ &+\lambda_R\sum_{k=2}^{\infty}\bar{\omega}_R^k(-1)^k\,\overline{\left(r_1(x)\frac{d}{dx}\right)^k x^m}_t\end{aligned} \tag{2.33}$$

3. WHITE POISSON NOISE IN A FINITE SYSTEM

In the previous section we have considered white Poisson external noise in the deterministic equation (2.16). This equation was obtained in the thermodynamic limit from the master equation (2.5). In this section we consider the same external noise but without going first to the thermodynamic limit. This is done by introducing random parameters at the level of the master equation (2.5). In this way we obtain a description of a finite system in the presence of external noise which takes into account intrinsic and external fluctuations simultaneously. It is obvious that this procedure is a purely phenomenological way of introducing external noise. The problem of deriving a master equation with stochastic transitions form first principles stands as a clear challenge. The introduction of random parameters can be made directly in (2.5) or equivalently in the equation (2.8) for the generating function or in equation (2.12) for the Poisson transform. A last possibility is to introduce random parameters in the equations for the moments obtained from (2.5) of (2.8)–(2.12). Each of these possibilities is more convenient for the calculation of different quantities in particular cases. For completeness we present below the general equivalent equations for the four approaches to the problem that we have mentioned.

In the same way than in Section 2 we assume that a parameter α in $Q(N)=Q_0^0(N)+\alpha Q_1(N)$ and a parameter β in $R(N)=R_0^0(N)+\beta R_1(N)$ become random functions of time as given in (2.17). This leads to stophastic transition probabilities

$$Q(N)=Q_0(N)+\xi_Q(t)\,Q_1(N) \tag{3.1}$$

$$R(N)=R_0(N)+\xi_R(t)\,R_1(N) \tag{3.2}$$

where

$$Q_0(N)=Q_0^0(N)+\bar{\alpha}Q_1(N) \tag{3.3}$$

$$R_0(N)=R_0^0(N)+\bar{\beta}R_1(N) \tag{3.4}$$

In the thermodynamic limit (3.1) and (3.2) reduce to (2.18) and (2.19), respectively. In the absence of external noise, the positivity of $Q(N)$ and $R(N)$ is in general guaranteed by the positivity of α and β. When these parameters become random functions of time, the positivity of $Q(N)$ and $R(N)$ require that α and β remain positive for all times. For white Poisson noise this is guaranteed whenever

$$\bar{\alpha}\geqslant\lambda_Q\bar{\omega}_Q \tag{3.5}$$

$$\bar{\beta}\geqslant\lambda_R\bar{\omega}_R \tag{3.6}$$

We will always assume that (3.5) and (3.6) are satisfied. We also note that α and β are taken as intensive quantities. The "external noise" character of $\xi_Q(t)$ and $\xi_R(t)$ implies that their parameters λ_Q, λ_R, ω_Q, and ω_R are independent of the system size: they have the same value as in the thermodynamic limit.

3.1. Master Equation

Substituting (3.1) and (3.2) in (2.5) we obtain a stochastic master equation

$$\frac{\partial P(N,t)}{\partial t}=[\Gamma_0+\Gamma_{1,Q}\xi_Q(t)+\Gamma_{1,R}\xi_R(t)]\,P(N,t) \tag{3.7}$$

where Γ_0 is the operator Γ defined in (2.5) with $Q(N)$, $R(N)$ replaced by $Q_0(N)$, $R_0(N)$ and

$$\Gamma_{1,Q}=(E^- -1)\,Q_1(N) \tag{3.8}$$

$$\Gamma_{1,R}=(E^+ -1)\,R_1(N) \tag{3.9}$$

Equation (3.5) is of the general form (2.26) with two independent white Poisson processes. From (2.28) we obtain

$$\frac{\partial \bar{P}(N,t)}{\partial t}=\Gamma_0\bar{P}(N,t)+\lambda_Q[\{e^{\omega_Q\Gamma_{1,Q}}\}_{\text{av}}-\bar{\omega}_Q\Gamma_{1,Q}-1]\,\bar{P}(N,t)$$
$$+\,\lambda_R[\{e^{\omega_R\Gamma_{1,R}}\}_{\text{av}}-\bar{\omega}_R\Gamma_{1,R}-1]\,\bar{P}(N,t) \tag{3.10}$$

$\bar{P}(N,t)$ is defined as the average of $P(N,t)$ over the realizations of $\xi_Q(t)$ and $\xi_R(t)$. It is the probability density which takes simultaneously into account intrinsic and external fluctuations. Equation (3.10) is the master equation for the joint description of the two types of fluctuations. When the values of ω are distributed with the exponential distribution (2.30), (3.10) becomes

$$\frac{\partial \bar{P}(N,t)}{\partial t}=(\Gamma_0-\lambda_Q\bar{\omega}_Q\Gamma_{1,Q}-\lambda_R\bar{\omega}_R\Gamma_{1,R})\,\bar{P}(N,t)$$
$$+\,\lambda_Q\bar{\omega}_Q\Gamma_{1,Q}\frac{1}{1-\bar{\omega}_Q\Gamma_{1,Q}}\,\bar{P}(N,t)$$
$$+\,\lambda_R\bar{\omega}_R\Gamma_{1,R}\frac{1}{1-\bar{\omega}_R\Gamma_{1,R}}\,\bar{P}(N,t) \tag{3.11}$$

It is obvious that if the master equation (3.10) is written in the form (2.1) in general it will involve nonvanishing transition probabilities $\bar{W}(N,N\pm n)$ for any n. [We denote by $\bar{W}$ the effective transition probabilities associated with the master equation (3.10).] A general expression for these transition probabilities is not easily obtained. For illustrative purposes we consider here the two cases of external noise which are more often studied in the literature. These are also the cases that we will consider in the examples of Section 4. The first case is that of additive external noise in which $Q_1(N)$ is independent of N. We take

$$Q_1(N) = aV \tag{3.12}$$

In the second case we consider a multiplicative external noise of the form

$$Q_1(N) = aN, \qquad R_1(N) = bN \tag{3.13}$$

In the first case (3.12) we have

$$\{e^{\omega_Q \Gamma_{1,Q}}\}_{\text{av}} \bar{P}(N, t) = \sum_{n=0}^{\infty} \{e^{-\omega_Q aV} \omega_Q^n\}_{\text{av}} \frac{a^n V^n}{n!} \bar{P}(N-n, t) \tag{3.14}$$

Substituting (3.14) in (3.10) the master equation can be written in the form (2.1) (with P and W replaced by $\bar{P}$ and $\bar{W}$, respectively) with the following averaged transition probabilities:

$$\bar{W}(N, N-1) = Q_0(N-1) + \lambda_Q aV\{\omega_Q(e^{-\omega_Q aV} - 1)\}_{\text{av}} \tag{3.15}$$

$$\bar{W}(N, N-n) = \lambda_Q\{\omega_Q^n e^{-\omega_Q aV}\}_{\text{av}} \frac{(aV)^n}{n!}, \qquad n > 1 \tag{3.16}$$

These transition probabilities take into account the effect of external additive noise (Q_1 constant) in a master equation formulation. The presence of external noise modifies the one-step transition probabilities $W(N, N-1)$ and introduces new nonvanishing transition probabilities $\bar{W}(N, N-n)$, $n > 1$. Given (3.3), (3.4), and (3.12), the requirements (3.5) and (3.6) are sufficient conditions which guarantee the positivity of $\bar{W}(N, N-1)$. The new transition probabilities $\bar{W}(N, N-n)$, $n > 1$ are always positive.

In the second case (3.13) we have

$$e^{\omega_Q \Gamma_{1,Q}} = e^{\omega_Q a[E^- - 1]N} = e^{-\omega_Q aN} \exp[(e^{\omega_Q a} - 1) E^- N] \tag{3.17}$$

$$e^{\omega_R \Gamma_{1,R}} = e^{\omega_R b[E^+ - 1]N} = e^{-\omega_R bN} \exp[(1 - e^{-\omega_R b}) E^+ N] \tag{3.18}$$

The second equalities in (3.17) and (3.18) are proved in Appendix B. They follow, respectively, from commutation properties of the operator $\omega_Q aN$ with $\omega_Q a[E^- - 1]N$ and $\omega_R bN$ with $\omega_R b[E^+ - 1]N$. With (3.17) and (3.18) we obtain

$$\{e^{\omega_Q \Gamma_{1,Q}}\}_{\text{av}} \bar{P}(N, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \{e^{-\omega_Q aN}(e^{\omega_Q a} - 1)^n\}_{\text{av}} \times (N-1) \cdots (N-n) \bar{P}(N-n, t) \tag{3.19}$$

$$\{e^{\omega_R \Gamma_{1,R}}\}_{\rm av}\, \bar{P}(N,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \{e^{-\omega_R b N}(1-e^{-\omega_R b})^n\}_{\rm av} \times (N+1)\cdots(N+n)\, \bar{P}(N+n,t) \tag{3.20}$$

Substituting (3.19) and (3.20) in (3.10) we obtain the master equation with the following averaged transition probabilities

$$\bar{W}(N,N-1) = Q_0(N-1) + \lambda_Q[\{e^{-\omega_Q a N}(e^{\omega_Q a}-1)\}_{\rm av} \times (N-1) - \bar{\omega}_Q a(N-1)] \tag{3.21}$$

$$\bar{W}(N,N+1) = R_0(N+1) + \lambda_R[\{e^{-\omega_R b N}(1-e^{-\omega_R b})\}_{\rm av} \times (N+1) - \bar{\omega}_R b(N+1)] \tag{3.22}$$

$$\bar{W}(N,N-n) = \lambda_Q[\{e^{-\omega_Q a N}(e^{a\omega_Q}-1)^n\}_{\rm av} \times (N-1)\cdots(N-n)/n!], \qquad n>1 \tag{3.23}$$

$$\bar{W}(N,N+n) = \lambda_R[\{e^{-\omega_R b N}(1-e^{-b\omega_R})^n\}_{\rm av} \times (N+1)\cdots(N+n)/n!], \qquad n>1 \tag{3.24}$$

In the same way that for the additive external noise case (3.12), the multiplicative external noise modifies the one-step transition probabilities and introduces new transition probabilities of n steps for all n. The difference with the additive case is that now $\bar{W}(N, N\pm n)$ depends also on N and not only on n. The same is true for the new term in $\bar{W}(N, N\pm 1)$. The requirements (3.5) and (3.6) guarantee the positivity of $\bar{W}(N, N\pm 1)$.

Finally we consider the Gaussian white noise limit of the white Poisson noise $\xi_Q(t)$ and $\xi_R(t)$. In this limit

$$\lambda_Q[\{e^{\omega_Q \Gamma_{1,Q}}\}_{\rm av} - \bar{\omega}_Q \Gamma_{1,Q} - 1] \to D_Q \Gamma_{1,Q}^2 \tag{3.25}$$

$$\lambda_R[\{e^{\omega_R \Gamma_{1,R}}\}_{\rm av} - \bar{\omega}_R \Gamma_{1,R} - 1] \to D_R \Gamma_{1,R}^2 \tag{3.26}$$

Substituting (3.25) and (3.26) in (3.10) we obtain an effective master equation for $\bar{P}(N,t)$ in the Gaussian white noise limit for $\xi_Q(t)$ and $\xi_R(t)$. This equation was directly obtained in Ref. 6 and also in Ref. 7 as the Gaussian white noise limit of the case in which $\xi_Q(t)$ and $\xi_R(t)$ are dichotomic Markov processes. In this limit the effective master equation (3.10) only has nonvanishing transition probabilities $\bar{W}(N\pm n, N)$ for

$n=1, 2$. We also remark that in this limit the positivity of the transition probabilities $\bar{W}(N \pm n, N)$ is not guaranteed since (3.5) and (3.6) cannot be satisfied.

3.2. Generating Function

A formulation of the problem equivalent to (3.10) is given in terms of an averaged generating function. Substituting (3.1), (3.2) in (2.8) we obtain a stochastic partial differential equation for $F(s, t)$:

$$\begin{aligned}\frac{\partial F(s,t)}{\partial t} = &\left[(s-1)\,Q_0\left(s\frac{\partial}{\partial s}\right) + \left(\frac{1}{s}-1\right) R_0\left(s\frac{\partial}{\partial s}\right)\right] F(s,t)\\ &+ (s-1)\,Q_1\left(s\frac{\partial}{\partial s}\right)\xi_Q(t)\,F(s,t)\\ &+ \left(\frac{1}{s}-1\right) R_1\left(s\frac{\partial}{\partial s}\right)\xi_R(t)\,F(s,t)\end{aligned} \tag{3.27}$$

This equation is again of the general form (2.26). Making use of the result (2.28) for the two independent processes $\xi_Q(t)$ and $\xi_R(t)$ we find

$$\begin{aligned}\frac{\partial \bar{F}(s,t)}{\partial t} = &\left[(s-1)\,Q_0\left(s\frac{\partial}{\partial s}\right) + \left(\frac{1}{s}-1\right) R_0\left(s\frac{\partial}{\partial s}\right)\right] \bar{F}(s,t)\\ &+ \lambda_Q\left(\left\{\exp\left[\omega_Q(s-1)\,Q_1\left(s\frac{\partial}{\partial s}\right)\right]\right\}_{\text{av}}\right.\\ &\left.- \bar{\omega}_Q(s-1)\,Q_1\left(s\frac{\partial}{\partial s}\right) - 1\right)\bar{F}(s,t)\\ &+ \lambda_R\left(\left\{\exp\left[\omega_R\left(\frac{1}{s}-1\right) R_1\left(s\frac{\partial}{\partial s}\right)\right]\right\}_{\text{av}}\right.\\ &\left.- \bar{\omega}_R\left(\frac{1}{s}-1\right) R_1\left(s\frac{\partial}{\partial s}\right) - 1\right)\bar{F}(s,t)\end{aligned} \tag{3.28}$$

$\bar{F}(s, t)$ is defined as the average of $F(s, t)$ over the realizations of $\xi_Q(t)$ and $\xi_R(t)$. It is the generating function associated with $\bar{P}(N, t)$

$$\bar{F}(s,t) = \sum_{N=0}^{\infty} s^N \bar{P}(N,t) \tag{3.29}$$

For the exponential distribution of ω (2.30), (3.28) reduces to

$$\begin{aligned}\frac{\partial \bar{F}(s,t)}{\partial t} = \Bigg\{ & (s-1)\left[Q_0\left(s\frac{\partial}{\partial s}\right) - \lambda_Q \bar{\omega}_Q Q_1\left(s\frac{\partial}{\partial s}\right)\right] \\ & + \left(\frac{1}{s}-1\right)\left[R_0\left(s\frac{\partial}{\partial s}\right) - \lambda_R \bar{\omega}_R R_1\left(s\frac{\partial}{\partial s}\right)\right]\Bigg\} \bar{F}(s,t) \\ & + \lambda_Q\left[\bar{\omega}_Q(s-1)\,Q_1\left(s\frac{\partial}{\partial s}\right)\frac{1}{1-\bar{\omega}_Q(s-1)\,Q_1[s(\partial/\partial s)]}\right]\bar{F}(s,t) \\ & + \lambda_R\left[\bar{\omega}_R\left(\frac{1}{s}-1\right)R_1\left(s\frac{\partial}{\partial s}\right)\frac{1}{1-\bar{\omega}_R(1/s-1)\,R_1[s(\partial/\partial s)]}\right]\bar{F}(s,t)\end{aligned} \tag{3.30}$$

The existence of nonvanishing transitions $N \to N \pm n$, $n > 1$, in (3.10) is here recognized by the appearance in the terms proportional to λ_Q and λ_R in (3.28) and (3.30) of contributions with a power in s given by the power of $s^{\pm n}\bar{F}(s,t)$, $n > 1$. In (2.10) only the power of $\bar{F}(s,t)$ and $s^{\pm 1}\bar{F}(s,t)$ exist.

3.3. Poisson Representation

We next consider the introduction of external noise in the equation (2.12) for the Poisson transform of $P(N, t)$. We assume that the fluctuating parameter α occurs only in the $m = i$ term in (2.9). That is,

$$\delta_i = \alpha V^{-i+1} \tag{3.31}$$

where the factor V^{-i+1} is introduced to conform the extensivity assumption for $Q(N)$. On the same grounds we take

$$v_j = \beta V^{-j+1} \tag{3.32}$$

Substituting (3.31), (3.32), and (2.17) in (2.9), (2.10) we obtain stochastic transition probabilities (3.1), (3.2) with

$$Q_1(N) = \frac{V\Omega_i(N)}{V^i}, \qquad R_1(N) = \frac{V\Omega_j(N)}{V^j} \tag{3.33}$$

With Eqs. (3.1), (3.2) and (3.33), (2.12) becomes the following stochastic partial differential equation for $f(a, t)$

$$\frac{\partial f(a,t)}{\partial t}=\left[\hat{Q}_0\left(a,\frac{\partial}{\partial a}\right)+\hat{R}_0\left(a,\frac{\partial}{\partial a}\right)\right]f(a,t)$$
$$+\left[\xi_Q(t)\,\hat{Q}_1\left(a,\frac{\partial}{\partial a}\right)+\xi_R(t)\,\hat{R}_1\left(a,\frac{\partial}{\partial a}\right)\right]f(a,t) \quad (3.34)$$

where

$$\hat{Q}_0\left(a,\frac{\partial}{\partial a}\right)=-\sum_{m=0}^{m_0}\hat{Q}^m\left(a,\frac{\partial}{\partial a}\right),\qquad \hat{R}_0\left(a,\frac{\partial}{\partial a}\right)=\sum_{l=1}^{l_0}\hat{R}^l\left(a,\frac{\partial}{\partial a}\right) \quad (3.35)$$

$\hat{Q}^m$ and $\hat{R}^l$ are as defined in (2.13), (2.14) but now with α replaced by $\bar{\alpha}$ in $\hat{Q}^i$ and β replaced by $\bar{\beta}$ in $\hat{R}^j$. The operators $\hat{Q}_1$, and $\hat{R}_1$ are

$$\hat{Q}_1\left(a,\frac{\partial}{\partial a}\right)=-V^{-i+1}\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^i a^i,$$
$$\hat{R}_1\left(a,\frac{\partial}{\partial a}\right)=V^{-j+1}\frac{\partial}{\partial a}\left(1-\frac{\partial}{\partial a}\right)^{j-1} a^j \quad (3.36)$$

We define $\bar{f}(a,t)$ as the average of $f(a,t)$ over the realizations of $\xi_Q(t)$ and $\xi_R(t)$. It is the Poisson transform of $\bar{P}(N,t)$. Equation (3.34) is again of the form (2.26). Its average can be taken with the help of (2.28). We obtain

$$\frac{\partial \bar{f}(a,t)}{\partial t}=\left[\hat{Q}_0\left(a,\frac{\partial}{\partial a}\right)+\hat{R}_0\left(a,\frac{\partial}{\partial a}\right)\right]\bar{f}(a,t)$$
$$+\lambda_Q\left[\left\{\exp\left[\omega_Q\hat{Q}_1\left(a,\frac{\partial}{\partial a}\right)\right]\right\}_{\rm av}-\bar{\omega}_Q\hat{Q}_1\left(a,\frac{\partial}{\partial a}\right)-1\right]\bar{f}(a,t)$$
$$+\lambda_R\left[\left\{\exp\left[\omega_R\hat{R}_1\left(a,\frac{\partial}{\partial a}\right)\right]\right\}_{\rm av}-\bar{\omega}_R\hat{R}_1\left(a,\frac{\partial}{\partial a}\right)-1\right]\bar{f}(a,t)$$
$$(3.37)$$

For the exponential distribution of ω (2.30), (3.37) reduces to

$$\frac{\partial \bar{f}(a,t)}{\partial t}=\left[\hat{Q}_0\left(a,\frac{\partial}{\partial a}\right)+\hat{R}_0\left(a,\frac{\partial}{\partial a}\right)\right]\bar{f}(a,t)$$
$$+\lambda_Q\left\{\left[1-\bar{\omega}_Q\hat{Q}_1\left(a,\frac{\partial}{\partial a}\right)\right]^{-1}-\bar{\omega}_Q\hat{Q}_1\left(a,\frac{\partial}{\partial a}\right)-1\right\}\bar{f}(a,t)$$
$$+\lambda_R\left\{\left[1-\bar{\omega}_R\hat{R}_1\left(a,\frac{\partial}{\partial a}\right)\right]^{-1}-\bar{\omega}_R\hat{R}_1\left(a,\frac{\partial}{\partial a}\right)-1\right\}\bar{f}(a,t) \quad (3.38)$$

The master equation (3.10), the equation for the generating function $\bar{F}(s,t)$, (3.28) and (3.37) give equivalent alternative formulations of a joint description of internal and external fluctuations. Equations (3.28) and (3.37) are quite similar at the formal level from a calculational point of view. Both have the advantage over (3.10) of being partial differential equations. Therefore it is easier in general to deal with them than with (3.10). In particular, the stationary solution of (3.30) and (3.38) can be found in several cases (see Section 4). The Poisson transform $\bar{f}(a, t)$ is more directly connected with the probability density of the process.[7]

3.4. Equations for the Moments

The simultaneous effect of internal and external fluctuations is clearly seen in the equations satisfied by the moments of the process $\overline{\langle N^m \rangle_t}$. By the bracket $\langle \cdots \rangle_t$ we indicate the average with $P(N, t)$:

$$\overline{\langle N^m \rangle_t} = \overline{\sum_{N=0}^{\infty} N^m P(N, t)} = \sum_{N=0}^{\infty} N^m \bar{P}(N, t) \tag{3.39}$$

The equations for $\overline{\langle N^m \rangle_t}$ can be calculated from (3.10), (3.29), or (3.37). It is interesting to derive these equations directly from the equations satisfied by $\langle N^m \rangle_t$ taking the average over the external noise. Here we present this derivation for the factorial moments $\overline{\langle \Omega_m(N) \rangle}$. The quantities $\Omega_m(N)$ were defined in (2.11).

The equation for $\langle \Omega_m(N) \rangle_t$ is obtained from (2.5). Substituting in this equation (3.1) and (3.2) we find

$$\begin{aligned}\frac{d}{dt} \langle \Omega_m(N) \rangle_t = {} & m\langle \Omega_{m-1}(N)\, Q_0(N) \rangle_t - m\langle \Omega_{m-1}(N-1)\, R_0(N) \rangle_t \\ & + m\langle \Omega_{m-1}(N)\, Q_1(N) \rangle_t\, \xi_Q(t) \\ & - m\langle \Omega_{m-1}(N-1)\, R_1(N) \rangle_t\, \xi_R(t)\end{aligned} \tag{3.40}$$

Equation (3.40) defines $\langle \Omega_m(N) \rangle_t$ as a functional of $\xi_Q(t)$ and $\xi_R(t)$. The average of (3.40) over the realizations of $\xi_Q(t)$ and $\xi_R(t)$ can be taken making use of the general formula (2.28). We obtain

$$\begin{aligned}\frac{d}{dt} \overline{\langle \Omega_m(N) \rangle_t} = {} & m\overline{\langle \Omega_{m-1}(N)\, Q_0(N) \rangle_t} - m\overline{\langle \Omega_{m-1}(N-1)\, R_0(N) \rangle_t} \\ & + \lambda_Q \sum_{n=2}^{\infty} \frac{\{\omega_Q^n\}_{\mathrm{av}}}{n!} \overline{\frac{\delta^{n-1}}{\delta \xi_Q(t)^{n-1}}\, m\langle \Omega_{m-1}(N)\, Q_1(N) \rangle_t} \\ & - \lambda_R \sum_{n=2}^{\infty} \frac{\{\omega_R^n\}_{\mathrm{av}}}{n!} \overline{\frac{\delta^{n-1}}{\delta \xi_R(t)^{n-1}}\, m\langle \Omega_{m-1}(N-1)\, R_1(N) \rangle_t}\end{aligned} \tag{3.41}$$

In order to get a more explicit form of (3.41) we need to specify $Q_1(N)$ and $R_1(N)$. We restrict ourselves to the two cases (3.12) and (3.13) analyzed in Section 3.1.

In the first case (3.12) we straightforwardly obtain

$$\frac{d}{dt}\langle\overline{\Omega_m(N)}\rangle_t = m\langle\overline{\Omega_{m-1}(N)\,Q_0(N)}\rangle_t - m\langle\overline{\Omega_{m-1}(N-1)\,R_0(N)}\rangle_t$$

$$+\lambda_Q \sum_{n=2}^{m}\binom{m}{n}\{\omega_Q^n\}_{\text{av}}(aV)^n\langle\overline{\Omega_{m-1}(N)}\rangle_t \tag{3.42}$$

In particular, for the first two moments and writing the equations in terms of $x=N/V$ we have

$$\frac{d}{dt}\langle\bar{x}\rangle_t = \langle\overline{q_0(x)}\rangle_t - \langle\overline{r_0(x)}\rangle_t \tag{3.43}$$

$$\frac{d}{dt}\langle\overline{x^2}\rangle_t = 2\langle\overline{x[q_0(x)-r_0(x)]}\rangle_t + \frac{1}{V}\langle\overline{q_0(x)+r_0(x)}\rangle_t$$

$$+\lambda_Q\{\omega_Q^2\}_{\text{av}}\,a^2 \tag{3.44}$$

where $q_0(x)$ and $r_0(x)$ have the same meaning as in (2.18)–(2.21). In the thermodynamic limit (3.42) reduces to a special case of (2.33). We next consider the case of multiplicative noise (3.13). We first analyze the term proportional to λ_R in (3.41). This term can be shown to be

$$\lambda_R[\{e^{-\omega_R bm}\}_{\text{av}} + \bar{\omega}_R bm - 1]\langle\overline{\Omega_m(N)}\rangle_t \tag{3.45}$$

The simplicity of the calculation of this term is due to the fact that the successive functional derivatives are proportional to $\langle\Omega_m(N)\rangle_t$ without involving any other factorial moment. This does not occur when calculating $(\delta^k/\delta\xi_Q^k(t))\langle\Omega_{m-1}(N)\,Q_1(N)\rangle_t$ and the calculation of the term proportional to λ_Q in (3.41) requires some more care. We first obtain from (3.40) that

$$\frac{\delta^{n-1}}{\delta\xi_Q^{n-1}(t)}\,m\langle\Omega_{m-1}(N)\,Q_1(N)\rangle_t = [am(1+D^-m)]^n\langle\Omega_m(N)\rangle_t \tag{3.46}$$

where we have introduced an operator D^- acting on $\varphi(m)$ as

$$D^-\varphi(m) \equiv \varphi(m-1) \tag{3.47}$$

Therefore,

$$\lambda_Q \sum_{n=2}^{\infty} \frac{\{\omega_Q^n\}_{\text{av}}}{n!} \frac{\delta^{n-1}}{\delta\xi_Q^{n-1}(t)} m\langle\overline{\Omega_{m-1}(N)\,Q_1(N)}\rangle_t$$
$$= \lambda_Q[\{e^{\omega_Q am(1+D^- m)}\}_{\text{av}} - \bar{\omega}_Q am(1+D^- m) - 1]\langle\overline{\Omega_m(N)}\rangle_t \tag{3.48}$$

This result can be made more explicit with the following operator identity:

$$e^{\omega_Q am(1+D^- m)} = e^{\omega_Q am} \exp\{(1-e^{-\omega_Q a})m\, D^- m\} \tag{3.49}$$

This relation is proved in Appendix B. It is a consequence of the commutation properties of $-a\omega_Q m$ with $a\omega_Q m(1+D^- m)$. Substituting (3.49) in (3.48) and operating with D^- we find the final expression for the term proportional to λ_Q in (3.41). Substituting such an expression and (3.45) in (3.41) we finally obtain the equation for $\langle\overline{\Omega_m(N)}\rangle_t$:

$$\frac{d}{dt}\langle\overline{\Omega_m(N)}\rangle_t = m\langle\overline{\Omega_{m-1}(N)\,Q_0(N)}\rangle_t - m\langle\overline{\Omega_{m-1}(N-1)\,R_0(N)}\rangle_t$$
$$+ \lambda_R[\{e^{-\omega_R bm}\}_{\text{av}} + \bar{\omega}_R bm - 1]\langle\overline{\Omega_m(N)}\rangle_t$$
$$+ \lambda_Q[(\{e^{\omega_Q am}\}_{\text{av}} - \bar{\omega}_Q am - 1)\langle\overline{\Omega_m(N)}\rangle_t$$
$$+ (\{e^{\omega_Q am}(1-e^{-\omega_Q a})\}_{\text{av}} - \bar{\omega}_Q a)\, m(m-1)\langle\overline{\Omega_{m-1}(N)}\rangle_t$$
$$+ \sum_{n=2}^{m-1} \frac{\{e^{\omega_Q am}(1-e^{-\omega_Q a})^n\}_{\text{av}}}{n!}$$
$$\times\, m(m-1)^2 \cdots (m-n+1)^2 (m-n)\langle\overline{\Omega_{m-1}(N)}\rangle_t] \tag{3.50}$$

In the thermodynamic limit (3.50) reduces to a particular case of (2.33). Of special relevance are the equations for the first two moments obtained from (3.50):

$$\frac{d\langle\bar{x}\rangle_t}{dt} = \langle\overline{q_0(x) - r_0(x)}\rangle_t + [\lambda_Q(\{e^{\omega_Q a}\}_{\text{av}} - \bar{\omega}_Q a - 1)$$
$$+ \lambda_R(\{e^{-\omega_R b}\}_{\text{av}} + \bar{\omega}_R b - 1)]\langle\bar{x}\rangle_t \tag{3.51}$$

$$\frac{d\langle\overline{x^2}\rangle_t}{dt} = 2\langle\overline{x[q_0(x)-r_0(x)]}\rangle_t + \frac{1}{V}\langle\overline{[q_0(x)+r_0(x)]}\rangle_t$$

$$+ \lambda_Q(\{e^{2\omega_Q a}\}_{\text{av}} - 2\bar{\omega}_Q a - 1)\langle\overline{x^2}\rangle_t$$

$$+ \lambda_R(\{e^{-2\omega_R b}\}_{\text{av}} - 2\bar{\omega}_R b - 1)\langle x^2\rangle_t$$

$$+ \frac{\lambda_Q}{V}(\{e^{2\omega_Q a}\}_{\text{av}} - \{e^{\omega_Q a}\}_{\text{av}} - \bar{\omega}_Q a)\langle\bar{x}\rangle_t$$

$$- \frac{\lambda_R}{V}(\{e^{-2\omega_R b}\}_{\text{av}} - \{e^{-\omega_R b}\}_{\text{av}} + \bar{\omega}_R b)\langle\bar{x}\rangle_t \qquad (3.52)$$

The two sets of equations (3.43), (3.44) and (3.51), (3.52) give some insight into the joint effect of internal and external fluctuations. Equations (3.43) and (3.51) coincide with the corresponding ones obtained in the thermodynamic limit directly from (2.29): there is no contribution from internal fluctuations in the equations for $\langle\bar{x}\rangle_t$ for the two examples of external noise that we consider. The interplay between internal and external fluctuations is seen in the equation for $\langle\overline{x^2}\rangle_t$. The two first terms on the right-hand side of (3.44) and (3.52) coincide. The first one gives the macroscopic equation and it is the only one that survives in the deterministic limit. The second one is a contribution from the internal fluctuations described by the master equation (2.1). The third and fourth terms in (3.44) and (3.52) are external noise contributions that already appear in the thermodynamic limit. They also follow from (2.29). In addition of these independent contributions from internal and external fluctuations the last two terms in (3.52) give contributions which come from the coupling of the two types of fluctuations. These contributions vanish in the thermodynamic limit and also in the limit of vanishing external noise. They are a genuine novel effect that only appears when both fluctuations are considered simultaneously. These "crossed-fluctuation" terms appear in our equations in spite of the implicit assumption of the independence of internal and external noise. We note that these terms do not exist for the additive external noise case (3.44). The crossed-fluctuation terms found here are not a peculiarity of the white Poisson noise that we have considered. A coupling effect of internal and external fluctuations also exists when the external noise is modeled by a dichotomic markov process[7,8] or in the Gaussian white noise limit of the theory.[6,8] The coupling effect seems to be an important physical consequence of any reasonable joint description of internal and external fluctuations. We remark that this coupling effect cannot be obtained if internal fluctuations are simply modeled by adding an independent gaussian white noise to the stochastic differential equation obtained from (2.16) and (2.18), (2.19), as it is often done.

3.5. Expansion Around the Thermodynamic Limit

In a macroscopic system in the presence of external noise, this is in general more important than the internal fluctuations which scale with the system size. The contribution of internal fluctuations can be calculated as a finite size effect by an expansion in powers of $1/V$ around the thermodynamic limit. We present here such an expansion to first order in $1/V$ starting from the effective master equation (3.10). New features that appear in the unified treatment of internal and external fluctuations such as the contributions which couple the two type of fluctuations show up in this expansion.

We define a probability density $\bar{P}(x, t)$ by

$$\bar{P}(x, t) = \bar{P}(N, t)V \tag{3.53}$$

Substituting $\partial/\partial N$ by $1/V(\partial/\partial x)$ in the operators $E^{\pm}$ and expanding to first order in $1/V$, the operators in (3.10) become

$$\Gamma_0 = -\frac{\partial}{\partial x}\left[q_0(x) - r_0(x)\right] + \frac{1}{2V}\frac{\partial^2}{\partial x^2}\left[q_0(x) + r_0(x)\right] \tag{3.54}$$

$$\Gamma_{1,Q} = \left(-\frac{\partial}{\partial x} + \frac{1}{2V}\frac{\partial^2}{\partial x^2}\right) q_1(x) \tag{3.55}$$

$$\Gamma_{1,R} = \left(\frac{\partial}{\partial x} + \frac{1}{2V}\frac{\partial^2}{\partial x^2}\right) r_1(x) \tag{3.56}$$

where the definition of q_0, r_0, q_1, and r_1 follows from (2.15), (3.1)–(3.4) and (2.18)–(2.21). Substituting (3.54)–(3.56) in (3.10) and keeping only terms of order $1/V$ we obtain

$$\frac{\partial \bar{P}(x, t)}{\partial t} = \left(L_0 + \frac{L_1}{V}\right)\bar{P}(x, t) + O(V^{-2}) \tag{3.57}$$

where L_0 is the operator defined in (2.29) and

$$L_1 = \frac{1}{2}\frac{\partial^2}{\partial x^2}[q_0(x) + r_0(x)]$$
$$+\frac{\lambda_Q}{2}\sum_{n=2}^{\infty}\frac{\{\omega_Q^n\}}{n!}\sum_{i=1}^{n}\left[-\frac{\partial}{\partial x}q_1(x)\right]^{n-i}\frac{\partial^2}{\partial x^2}q_1(x)\left[-\frac{\partial}{\partial x}q_1(x)\right]^{i-1}$$
$$+\frac{\lambda_R}{2}\sum_{n=2}^{\infty}\frac{\{\omega_R^n\}}{n!}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x}r_1(x)\right]^{n-i}\frac{\partial^2}{\partial x^2}r_1(x)\left[\frac{\partial}{\partial x}r_1(x)\right]^{i-1} \qquad (3.58)$$

When the quantities q_0, r_0, q_1, and r_1 contain terms proportional to V^{-n}, there are terms kept in (3.58) which are of order V^{-2} or higher. A consistent expansion in V^{-1} is easily rearranged in each particular case.

There exist two interesting limits of (3.58). In the limit $V \to \infty$ we recover Eq. (2.29) in which internal fluctuations are neglected and $\bar{P}(x, t) = P(x, t)$. In the limit in which external noise is neglected, $\lambda_Q\{\omega_Q^n\}_{\text{av}} \to 0$, $\lambda_R\{\omega_R^n\}_{\text{av}} \to 0$ for all n, we obtain a Focker–Planck description of internal noise. This description follows from a formal expansion in powers of $1/V$ of the master equation (2.1). Our result in this limit coincides with the general formal result of Horsthemke and Brenig.[22] (For a discussion and clarification of the last result see Ref. 5). We also note that the ordinary system size expansion of the master equation[3,4] cannot be straightforwardly applied to our case because for $V \to \infty$ we do not have a well-defined trajectory to expand around.[4]

Besides the independent contributions from internal fluctuations and external noise which appear in the two limits mentioned above, there is an additional contribution given by the last two terms in (3.58). These terms are a result of the coupling of internal and external fluctuations and give rise to the crossed-fluctuation terms discussed in Section 3.4 for particular cases of $q_1(x)$ and $r_1(x)$. The general form of these terms for a large but finite system follows from (3.58).

It could be argued that (3.57) is not of great practical interest for the calculation of $\bar{P}(x, t)$. First it contains in general derivatives of all orders with respect to x. Second, the validity of the formal expansion in $1/V$ can be questioned when used to obtain a detailed result for $\bar{P}(x, t)$. Nevertheless (3.57) is extremely useful to obtain information on the main statistical features of the system, in particular on the behavior of the low-

[4] For an overview of different attitudes with respect to the thermodynamic limit of the master equation in the absence of external noise see the discussion following the paper by R. Graham in Ref. 23.

order moments of $\bar{P}(x, t)$. The equations that follow from (3.57) for the first two moments are

$$\frac{d\langle \bar{x}\rangle_t}{dt} = \langle \overline{q_0(x) - r_0(x)}\rangle_t + \lambda_Q \sum_{n=2}^{\infty} \frac{\{\omega_Q^n\}_{\text{av}}}{n!} \left\langle \overline{q_1(x)\left[\frac{\partial}{\partial x} q_1(x)\right]^{n-1}} \right\rangle_t$$
$$+ \lambda_R \sum_{n=2}^{\infty} \frac{\{\omega_R^n\}_{\text{av}}}{n!} (-1)^n \left\langle \overline{r_1(x)\left[\frac{\partial}{\partial x} r_1(x)\right]^{n-1}} \right\rangle_t + \frac{\lambda_Q}{2V} \sum_{n=2}^{\infty} \frac{\{\omega_Q^n\}_{\text{av}}}{n!}$$
$$\times \sum_{i=1}^{n} \left\langle \overline{q_1(x)\left[\frac{\partial}{\partial x} q_1(x)\right]^{i-1} \frac{\partial^2}{\partial x^2} q_1(x) \left[\frac{\partial}{\partial x} q_1(x)\right]^{n-i-1}} \right\rangle_t$$
$$+ \frac{\lambda_R}{2V} \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\{\omega_R^n\}_{\text{av}}}{n!}$$
$$\times \sum_{i=1}^{n-1} \left\langle \overline{r_1(x)\left[\frac{\partial}{\partial x} r_1(x)\right]^{i-1} \frac{\partial^2}{\partial x^2} r_1(x) \left[\frac{\partial}{\partial x} r_1(x)\right]^{n-i-1}} \right\rangle_t \tag{3.59}$$

$$\frac{d\langle \overline{x^2}\rangle_t}{dt} = 2\langle \overline{x[q_0(x) - r_0(x)]}\rangle_t + \frac{1}{V} \langle \overline{q_0(x) + r_0(x)}\rangle_t$$
$$+ 2\lambda_Q \sum_{n=2}^{\infty} \frac{\{\omega_Q^n\}_{\text{av}}}{n!} \left\langle \overline{q_1(x)\left[\frac{\partial}{\partial x} q_1(x)\right]^{n-2} \frac{\partial}{\partial x} q_1(x)x} \right\rangle_t$$
$$+ 2\lambda_R \sum_{n=2}^{\infty} (-1)^n \frac{\{\omega_Q^n\}_{\text{av}}}{n!} \left\langle \overline{r_1(x)\left[\frac{\partial}{\partial x} r_1(x)\right]^{n-2} \frac{\partial}{\partial x} r_1(x)x} \right\rangle_t$$
$$+ \frac{\lambda_Q}{V} \sum_{n=2}^{\infty} \frac{\{\omega_Q\}^n}{n!} \left(\left\langle \overline{q_1(x)\left[\frac{\partial}{\partial x} q_1(x)\right]^{n-1}} \right\rangle_t \right.$$
$$+ \left\langle \overline{q_1(x)\left[\frac{\partial}{\partial x} q_1(x)\right]^{n-2} \frac{\partial^2}{\partial x^2} q_1(x)x} \right\rangle_t$$
$$\left. + \sum_{i=1}^{n-2} \left\langle \overline{q_1(x)\left[\frac{\partial}{\partial x} q_1(x)\right]^{i-1} \frac{\partial^2}{\partial x^2} q_1(x) \left[\frac{\partial}{\partial x} q_1(x)\right]^{n-i-2} \frac{\partial}{\partial x} q_1(x)x} \right\rangle_t \right)$$
$$+ \frac{\lambda_R}{V} \sum_{n=2}^{\infty} \frac{\{\omega_R^n\}_{\text{av}}}{n!} (-1)^{n-1} \left(\left\langle \overline{r_1(x)\left[\frac{\partial}{\partial x} r_1(x)\right]^{n-1}} \right\rangle_t \right.$$
$$+ \left\langle \overline{r_1(x)\left[\frac{\partial}{\partial x} r_1(x)\right]^{n-2} \frac{\partial^2}{\partial x^2} r_1(x)x} \right\rangle_t$$
$$\left. + \sum_{i=1}^{n-2} \left\langle \overline{r_1(x)\left[\frac{\partial}{\partial x} r_1(x)\right]^{i-1} \frac{\partial^2}{\partial x^2} r_1(x) \left[\frac{\partial}{\partial x} r_1(x)\right]^{n-i-2} \frac{\partial}{\partial x} r_1(x)x} \right\rangle_t \right) \tag{3.60}$$

In the two particular cases of external noise (3.12) and (3.13) considered in Section 3.4, the two equations (3.59) and (3.60) reproduce exactly equations (3.43), (3.44), (3.51), and (3.52). In these cases higher-order terms in the expansion in $1/V$ of (3.57) do not contribute to the equations for $\langle \bar{x} \rangle_t$ and $\langle \overline{x^2} \rangle_t$.

In the general case we have in the equation for $\langle \bar{x} \rangle_t$ three different types of contributions. The first term on the right-hand side of (3.59) gives the deterministic limit. The second and third terms are contributions from external noise already present in the thermodynamic limit. They also follow from (2.29) and they vanish for additive external noise. There is no independent contribution from internal fluctuations. The last two terms in (3.59) are crossed-fluctuation terms. They only appear if $q_1(x)$ or $r_1(x)$ is a nonlinear function of x. In Eq. (3.60) for $\langle \overline{x^2} \rangle_t$, the first term on the right-hand side gives the deterministic limit and the second one the independent contribution from internal fluctuations. The third and fourth terms are the contribution from external noise in the thermodynamic limit. They do not vanish even if $q_1(x)$ and $r_1(x)$ are constant. The last two terms are the crossed-fluctuation terms which in general only vanish for external additive noise.

4. EXAMPLES

4.1. Poisson Counting Process

As a first simple example we consider internal fluctuations described by a Poisson counting process[2] with parameter αV. This is defined by a one-step master equation (2.1) with

$$Q(N) = \alpha V \tag{4.1}$$

$$R(N) = 0 \tag{4.2}$$

and initial condition

$$P(N, 0) = \delta_{N,0} \tag{4.3}$$

The solution of (2.5) with (4.1)–(4.3) is a Poisson distribution

$$P(N, t) = \frac{(\alpha V t)^N}{N!} e^{-\alpha V t} \tag{4.4}$$

We now imagine that the parameter α becomes a random function of time like in (2.17). The fluctuations $\xi(t)$ are given by a white Poisson noise with parameter λ and $\rho(\omega) = \delta(\omega - \omega_0)$. A joint description of the two types of fluctuations considered is equivalently given by Eq. (3.10) for the averaged probability density $\bar{P}(N, t)$, Eq. (3.28) for the generating function $\bar{F}(s, t)$ or the Poisson representation equation (3.37). In this example (3.28) becomes

$$\frac{\partial \bar{F}(s, t)}{\partial t} = (s-1)\,\bar{\alpha} V \bar{F}(s, t) + \lambda[e^{\omega_0 V(s-1)} - \omega_0 V(s-1) - 1]\,\bar{F}(s, t) \tag{4.5}$$

whose solution is

$$\bar{F}(s, t) = \exp\{(s-1)(\bar{\alpha} V - \lambda\omega_0 V)t + \lambda t(e^{\omega_0 V(s-1)} - 1)\} \tag{4.6}$$

$\bar{F}(s, t)$ contains all the relevant statistical properties of the model. In particular the averaged probability density is obtained as

$$\bar{P}(N, t) = \frac{1}{N!} \frac{\partial^N \bar{F}(s, t)}{\partial s^N}\bigg|_{s=0} = \frac{e^{-\lambda t}}{N!} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left[\left(\bar{\alpha} V - \lambda\omega_0 V + \frac{k\omega_0 V}{t}\right) t\right]^N$$
$$\times \exp\left\{-\left[\bar{\alpha} V - \lambda\omega_0 V + \frac{k\omega_0 V}{t}\right] t\right\} \tag{4.7}$$

The condition $\bar{\alpha} \geqslant \lambda\omega_0$ guarantees the positivity of $\bar{P}(N, t)$ for all N and all t. Comparison of (4.4) with (4.7) shows how the Poisson distribution (4.4) is modified by Poisson white noise fluctuations of α. The probability distribution is now a superposition of Poisson distributions (with time-dependent parameter) weighted by another Poisson distribution. It is illustrative to see this modification at the level of the Poisson representation. We obviously have

$$f(a, t) = \delta(a - \alpha V t) \tag{4.8}$$

In this example (3.37) becomes

$$\frac{\partial \bar{f}(a, t)}{\partial t} = -\frac{\partial}{\partial a}(\bar{\alpha} V - \lambda\omega_0 V)\bar{f}(a, t) + \lambda(e^{\omega_0 V(\partial/\partial a)} - 1)\bar{f}(a, t) \tag{4.9}$$

This is easily solved by a Fourier transformation giving

$$\bar{f}(a, t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\, \delta\left[a - \left(\bar{\alpha} V - \lambda\omega_0 V + \frac{k\omega_0 V}{t}\right) t\right] \tag{4.10}$$

The fluctuations of α lead to the replacement of (4.8) by a superposition of δ functions with shifted arguments.

The factorial moments of (4.7) are given by

$$\langle\overline{\Omega_m(N)}\rangle_t = e^{-t\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left[\left(\alpha V - \lambda \omega_0 V + \frac{k\bar{\alpha} V}{t} \right) t \right]^m \tag{4.11}$$

The mean value $\langle \bar{N} \rangle_t$ coincides with that of the original process (4.4)

$$\langle \bar{N} \rangle_t = V\bar{\alpha} t \tag{4.12}$$

This is in agreement with the general equation (3.43). The non-Poissonian character of (4.7) is shown in the second moment:

$$\langle \bar{N}^2 \rangle_t - \langle \bar{N} \rangle_t^2 = \langle \bar{N} \rangle_t + \lambda \omega_0^2 V^2 t \tag{4.13}$$

The last term in (4.13) is the non-Poissonian contribution.

4.2. Creation and Annihilation Process with Source

As a second example we consider fluctuations of internal origin modeled by the one-step master equation (2.1) with transition probabilities

$$Q(N) = \alpha V + \gamma N \tag{4.14}$$

$$R(N) = \beta N \tag{4.15}$$

where V is the volume of the system. This is a model used to study many different nonequilibrium physical systems, for example, maser amplification,[2,17] a point nuclear reactor model,[19] and a chemical model.[18] For maser amplification N is the number of quanta in a given mode of the electromagnetic field, βN is the rate at which quanta disappear through the walls, γN the rate of stimulated emission, and αV the rate of input of quanta into the system. In the point nuclear reactor model, N is the number of neutrons at time t, β the capture rate per neutron, γ the fission rate per neutron, and α the source event rate per neutron per unit volume. The chemical model described by (4.14), (4.15) is given by the reactions

$$A + X \underset{k_1'}{\overset{k_1}{\rightleftharpoons}} C$$

$$B + X \xrightarrow[k_2]{} 2X$$

N is here the number of molecules of the species X and $\alpha = k_1' c$, $\gamma = k_2 b$, $\beta = k_1 a$, where a, b, and c are, respectively, the concentrations of the species A, B, C which are kept fixed.

The time-dependent properties of this model can be calculated exactly. The generating function $F(s, t)$ is given in Ref. 21 and the equation for the Poisson transform (2.12) is solved in Ref. 18. Here we restrict ourselves to the stationary properties. A stationary solution only exists for $\beta-\gamma>0$. In this case[21]

$$F_{st}(s)=\left(\frac{\beta-\gamma}{\beta-s\gamma}\right)^{V\alpha/\gamma} \tag{4.16}$$

or equivalently

$$P_{st}(N)=\frac{1}{N!}\left(\frac{\beta-\gamma}{\beta}\right)^{V\alpha/\gamma}\left(\frac{\gamma}{\beta}\right)^{N}\left(\frac{V\alpha}{\gamma}\right)_{N} \tag{4.17}$$

where $(x)_N=x(x+1)\cdots(x+N-1)$. We are here considering open systems and (4.17) represents a nonequilibrium steady state. The factorial moments of the distribution are

$$\langle\Omega_m(N)\rangle_{\mathrm{st}}=\left(\frac{\gamma}{\beta-\gamma}\right)^{m}\left(\frac{\alpha V}{\gamma}\right)_{m} \tag{4.18}$$

In particular, for the mean value and relative fluctuations of the intensive variable $x=N/V$ we have

$$\langle x\rangle_{\mathrm{st}}=\frac{\alpha}{\beta-\gamma} \tag{4.19}$$

$$\frac{(\Delta x)^2}{\langle x\rangle^2}=\frac{\langle x^2\rangle-\langle x\rangle^2}{\langle x\rangle^2}=\frac{\beta}{\alpha V} \tag{4.20}$$

The maximum of the stationary distribution given by $P_{\mathrm{st}}(N)=P_{\mathrm{st}}(N+1)$ is found at

$$x_{\max}=\frac{\alpha}{\beta-\gamma}-\frac{\beta}{V(\beta-\gamma)} \tag{4.21}$$

The above results characterize the stationary properties in a finite volume V. In the thermodynamic limit $V\to\infty$, $\langle x\rangle_{\mathrm{st}}=x_{\max}$ and the fluctuations around $\langle x\rangle_{\mathrm{st}}$ vanish. In this limit the system is described by the deterministic equation

$$\frac{dx}{dt}=-(\beta-\gamma)x+\alpha \tag{4.22}$$

The stationary solution of (4.22) for $\beta-\gamma>0$ coincides with (4.19). For $\beta-\gamma<0$, (4.22) leads to explosion of x.

In the following we analyze the effect of randomness in the control parameters of the system. We discuss separately the consequences of α, β, and γ fluctuations. General results when α and β or α and γ fluctuate simultaneously are given in the previous section.

4.2.1. External Noise in the Source Parameter. Here we take $\alpha \to \bar{\alpha}+\xi_Q(t)$, where $\xi_Q(t)$ is a Poisson white noise with parameter λ and an exponential distribution $\rho(\omega)$ (2.30). In our model system α is a positive definite quantity and therefore we require that $\bar{\alpha} \geqslant \lambda\bar{\omega}$.

4.2.1.1. Thermodynamic Limit. We first discuss the effect of external noise in the thermodynamic limit (4.22). Replacing α by $\bar{\alpha}+\xi_Q(t)$ in (4.22) we obtain a stochastic differential equation whose stationary distribution $P_{\mathrm{st}}(x)$ is obtained from (2.31). In this model the process x has a boundary at

$$x=x_0=\frac{\bar{\alpha}-\lambda\bar{\omega}}{\beta-\gamma} \tag{4.23}$$

The formal solution of (2.31) gives two normalizable stationary distributions when $\beta-\gamma>0$. The first is defined in the interval $(0, x_0)$ and the second in (x_0, ∞). From the stochastic differential equation we directly find that

$$\bar{x}_{\mathrm{st}}=\frac{\bar{\alpha}}{\beta-\gamma}>x_0 \tag{4.24}$$

This implies that the solution defined in $(0, x_0)$ has to be discarded. Therefore a stationary distribution exists for $\beta-\gamma>0$. It is defined in (x_0, ∞) and it is given by

$$P_{\mathrm{st}}(x)=Ne^{-x/\bar{\omega}}\{(\beta-\gamma)x-(\bar{\alpha}-\lambda\bar{\omega})\}^{\lambda/(\beta-\gamma)-1} \tag{4.25}$$

where the normalization constant N is

$$N=\frac{1}{\Gamma(\lambda/(\beta-\gamma))}\exp\left\{\frac{\bar{\alpha}-\lambda\bar{\omega}}{\bar{\omega}(\beta-\gamma)}\right\}\bar{\omega}^{-\lambda/(\beta-\gamma)}(\beta-\gamma)^{1-\lambda/(\beta-\gamma)} \tag{4.26}$$

A comparison of (4.17) and (4.25) is given in Fig. 1. The fact that $P_{\mathrm{st}}(x)$ is defined in (x_0, ∞) can be physically understood recalling that $\xi_Q(t)$ is bounded from below, $\xi_Q(t) \geqslant -\lambda\bar{\omega}$, but not from above. This leads to the inaccessibility of values of $x<x_0$, but there is no restriction at large x.

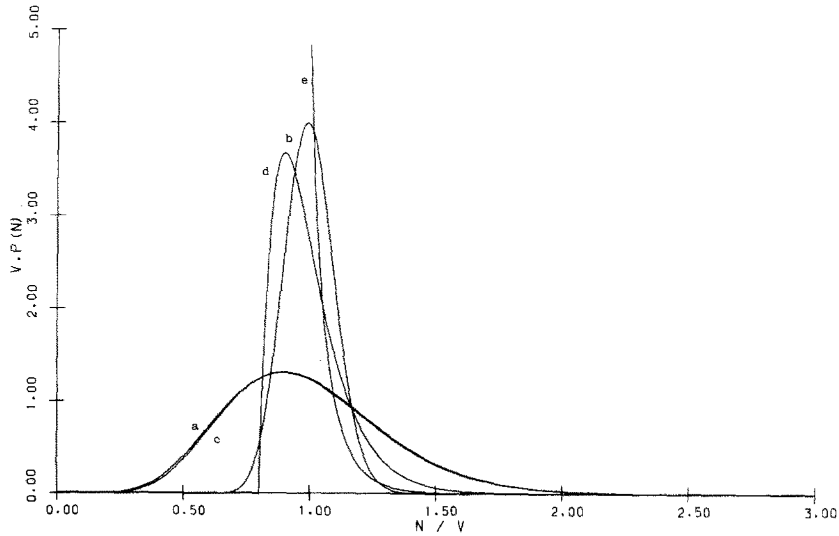

Fig. 1. Plot of different stationary probabilities comparing the effect of finite volume alone [Eq. (4.17)] with the effect of external noise in the source parameter α [Eq. (4.25)] in the thermodynamic limit. Values of the parameters: $\alpha = 0.1$, $\beta = 1$, $\gamma = 0.9$. Internal fluctuations (a) $V = 100$, (b) $V = 1000$. External fluctuations $\bar{\omega} = 0.1$, (c) $\lambda = 1$, (d) $\lambda = 0.2$, (e) $\lambda = 0.05$.

All the moments of (4.25) exist for $\beta - \gamma > 0$. They can be calculated from (4.25) or from the corresponding particular case of (2.33)

$$\bar{x}_{st}^m = \sum_{k=0}^{m} \binom{m}{k} \bar{\omega}^k x_0^{m-k} \left(\frac{\lambda}{\beta - \gamma}\right)_k \tag{4.27}$$

The relative fluctuation is

$$\frac{\overline{x_{st}^2} - \bar{x}_{st}^2}{\bar{x}_{st}^2} = \frac{\lambda \bar{\omega}^2}{\bar{\alpha}^2} (\beta - \gamma) \tag{4.28}$$

The mean value (4.24) coincides with the deterministic steady state and (4.28) measures the fluctuations around $\bar{x}_{st}$ due to the randomness of α.

A more closed inspection of (4.25) reveals a new feature caused by the external noise (Fig. 1). When $\beta - \gamma < \lambda$, $P_{st}(x_0) = 0$ and $P_{st}(x)$ has a single maximum at

$$x_{max} = \frac{\bar{\alpha}}{\beta - \gamma} - \bar{\omega} \tag{4.29}$$

In this situation $x_{\max}$ is shifted from $\bar{x}$ due to the randomness of α. A similar shift is caused by internal fluctuations when α is a constant parameter [compare (4.19) and (4.21)]. A different situation occurs for $\beta-\gamma>\lambda$. We then have that $P_{st}(x_0)=\infty$ and there is no relative extremum of $P_{st}(x)$. This difference appears as a new kind of transition at $\beta-\gamma=\lambda$. Such changes of the stationary distribution induced by changes in the parameters of the external noise have been widely discussed in recent years.[1,24] For Gaussian white external noise these changes have been in general associated with a multiplicative character of the noise. The external fluctuations of the source parameter have an additive character but they are not Gaussian. In the Gaussian white noise limit the same results (4.24) and (4.28) are found for the mean value and relative fluctuations, but no qualitative change of $P_{st}(x)$ exists for any value of the Gaussian white noise intensity $D=\lambda\bar{\omega}^2$. The transition found here with a change of P_{st} at the boundary $x=x_0$ is reminiscent of the transitions found when external noise is modeled by a dichotomic Markov process.[1] They are also associated with changes of $P_{st}(x)$ at the boundaries of the process. These type of transitions seem to be characteristic of external noise with bounded realizations. In our case $\xi_Q(t)$ is bounded from below and it evolves in a characteristic time λ^{-1}. When $\xi_Q(t)$ is slow in comparison with the deterministic evolution, that is, $\lambda^{-1}>(\beta-\gamma)^{-1}$, the process $x(t)$ approaches x_0 which is the steady state of the deterministic dynamics (4.22) with α replaced by $\bar{\alpha}-\lambda\bar{\omega}$. The value $\bar{\alpha}-\lambda\bar{\omega}$ is the lower bound for $\bar{\alpha}+\xi_Q(t)$. When $\xi_Q(t)$ becomes a fast process, that is, $\lambda^{-1}<(\beta-\gamma)^{-1}$, the competition between the deterministic dynamics and the fast-driving stochastic force moves the stationary distribution away from the boundary at $x=x_0$.

4.2.1.2. Finite System. Setting $\alpha=\bar{\alpha}+\xi_Q(t)$ in the transition probability (4.14) we are led to the joint description of fluctuations discussed in Section 3. In this particular example we have

$$Q_0(N)=\bar{\alpha}V+\gamma N, \qquad Q_1(N)=V \tag{4.30}$$

$R_0(N)$ is given by (4.15) and $R_1(N)=0$. We have as well $\lambda_Q=\lambda$, $\bar{\omega}_Q=\bar{\omega}$, $\lambda_R=0$.

The transition probabilities $\bar{W}(N, N\pm n)$ are given by (3.15), (3.16) with the specifications above and $a=1$. The original one-step process becomes in the presence of α fluctuations a process with nonvanishing transition probabilities with creation of n particles for all n.

The stationary properties of the system can be calculated through the stationary averaged generating function $\bar{F}_{st}(s)$. Equation (3.30) becomes here

$$\frac{\partial \bar{F}(s,t)}{\partial t} = [\beta(1-s) + \gamma s(s-1)] \frac{\partial \bar{F}(s,t)}{\partial s} + \left[\bar{\alpha} V(s-1) + \lambda \left(\frac{V\bar{\omega}(s-1)}{1 - V\bar{\omega}(s-1)} - V\bar{\omega}(s-1) \right) \right] \bar{F}(s,t) \tag{4.31}$$

whose normalized $\bar{F}(s=1)=1$ stationary solution is

$$\bar{F}_{st}(s) = \left(\frac{\beta - \gamma}{\beta - \gamma s} \right)^{(\bar{\alpha} - \lambda\bar{\omega})V/\gamma + \theta} \left[\frac{1}{1 - V\bar{\omega}(s-1)} \right]^{-\theta} \tag{4.32}$$

where

$$\theta = \frac{\lambda \bar{\omega} V}{\gamma - \bar{\omega} V(\beta - \gamma)} \tag{4.33}$$

The stationary probability distribution can be calculated from (4.32). It can be written as

$$\bar{P}_{st}(N) = \frac{1}{N!} \frac{d^N \bar{F}_{st}}{ds^N}\bigg|_{s=0} = \frac{1}{N!} \left(\frac{\beta}{\beta - \gamma} \right)^{-[V(\bar{\alpha} - \lambda\bar{\omega})/\gamma] - \theta} (1 + \bar{\omega} V)^{\theta} \left(\frac{\gamma}{\beta} \right)^{N} \times \left[\frac{V(\bar{\alpha} - \lambda\bar{\omega})}{\gamma} \right]_N F\left(-N, -\theta, \frac{V(\bar{\alpha} - \lambda\bar{\omega})}{\gamma}; 1 - \frac{\beta\bar{\omega} V}{\gamma(1 + \bar{\omega} V)} \right) \tag{4.34}$$

where $F(a, b, c; z)$ is the hypergeometric function.[25]

We note that a stationary state does not exist for $\beta - \gamma < 0$. In that situation $F_{st}(s)$ in (4.32) is not an analytic function for $|s| < 1$ because it has a pole at $s = \beta/\gamma < 1$. It can be checked that $\bar{P}_{st}(N)$ as given by (4.34) is a positive definite quantity under the additional requirement of $\bar{\alpha} > \lambda\bar{\omega}$. In comparison with the solution in the thermodynamic limit (4.25), we observe that internal fluctuations destroy the barriers found in that limit so that (4.34) is defined for all values of $N \geq 0$.

The combined effect of internal and external fluctuations in the form $\bar{P}_{st}(N)$ is shown in Figs. 2–4 where $\bar{P}_{st}(N)$ is plotted for different values of the parameters V and λ (see Appendix C). The parameters V and λ control the strength of these two types of fluctuations. As noted above, the boundary at x_0 disappears in the presence of internal fluctuations. As a consequence, the qualitative change in the form of P_{st} found in the thermodynamic limit at $\beta - \gamma = \lambda$ is smeared out in a finite system. Nevertheless, for large enough V a significant difference between $\beta - \gamma > \lambda$ and $\beta - \gamma < \lambda$ persists in the height of $\bar{P}_{st}(N)$ at $N = N_{max}$. As the system size becomes small, internal fluctuations dominate the system and no peculiar behavior is found at $\beta - \gamma = \lambda$. On the other hand for a fixed value

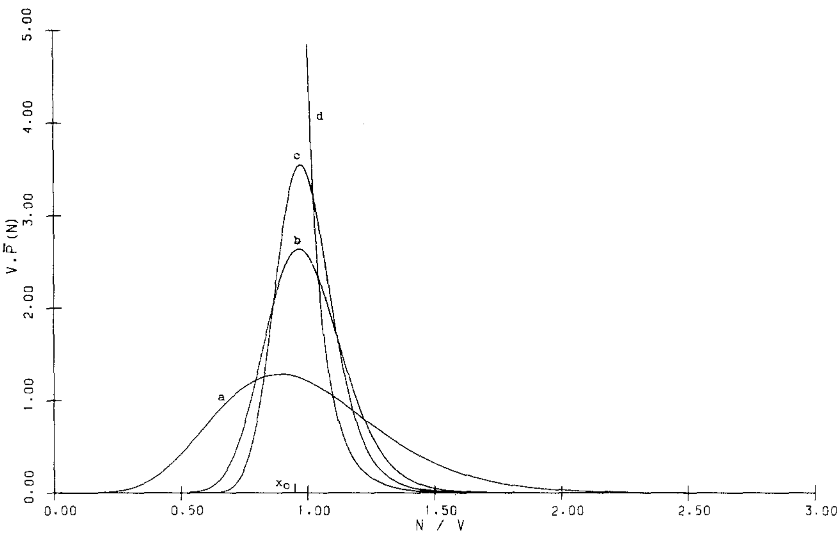

Fig. 2. Plot of the stationary probability (4.34) for different values of V and $\beta-\gamma>\lambda$. $\bar{\alpha}=0.1$, $\beta=1$, $\gamma=0.9$, $\bar{\omega}=0.1$, $\lambda=0.05$; (a) $V=100$, (b) $V=500$, (c) $V=1000$, (d) $V=\infty$. $x_0=0.95$.

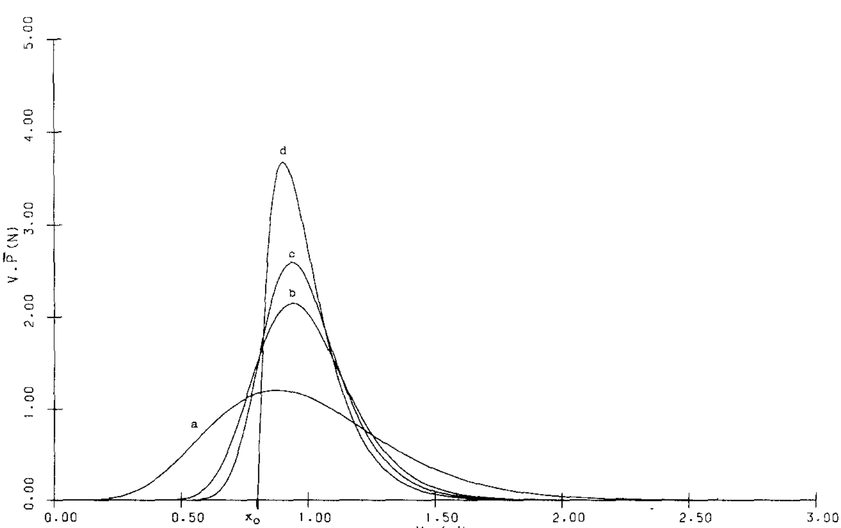

Fig. 3. Same as in Fig. 2 but with $\lambda=0.2$ $(\beta-\gamma<\lambda)$. $x_0=0.8$.

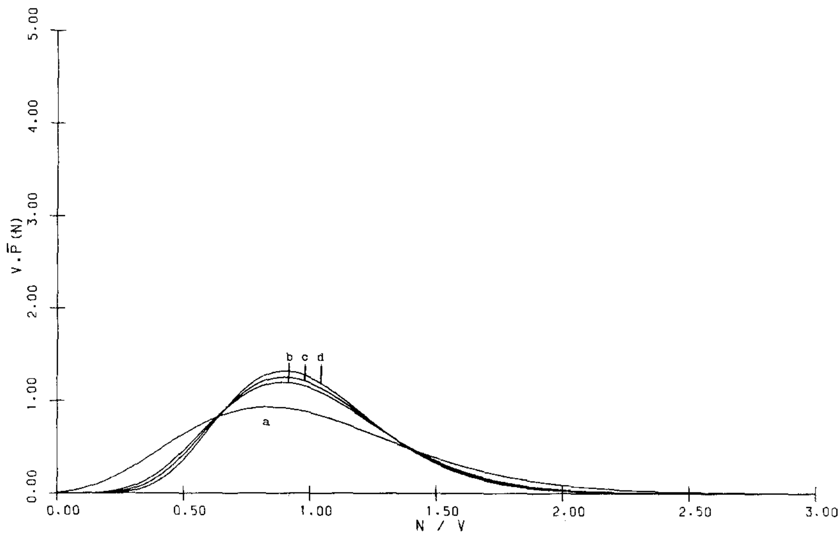

Fig. 4. Same as in Fig. 2 but with $\lambda = 1$ $(\beta - \gamma < \lambda)$. $x_0 = 0$.

of V, increasing the value of $\lambda > \beta - \gamma$, $\bar{P}_{st}(N)$ moves to smaller values of N/V. This is due to the fact that increasing λ implies that $\alpha = \bar{\alpha} + \xi_Q(t)$ takes smaller values because $\xi_Q(t) \geqslant -\lambda\bar{\omega}$.

It is illustrative to see how the result (4.34) is reobtained making use of the Poisson representation of the master equation. Equation (3.38) becomes in this example

$$\frac{\partial \bar{f}(a,t)}{\partial t} = \left\{ -\frac{\partial}{\partial a}\left[(\bar{\alpha} - \lambda\bar{\omega})V + (\gamma - \beta)a\right] + \gamma \frac{\partial^2}{\partial a^2} a + \lambda\left[\frac{1}{1 + \bar{\omega}V(\partial/\partial a)} - 1\right]\right\} \bar{f}(a,t) \tag{4.35}$$

A stationary solution of (4.35) can be found with the transformation

$$\bar{f}_{st}(a) = e^{-a/\bar{\omega}V} U(y) \tag{4.36}$$

$$y = \frac{\lambda a}{\gamma \theta} \tag{4.37}$$

$U(y)$ satisfies Kummer's equation[25] in the form

$$yU''(y) + \left[2 - \frac{(\bar{\alpha} - \lambda\bar{\omega})V}{\gamma} - y\right] U'(y) - (1 + \theta)\, U(y) = 0 \tag{4.38}$$

The normalized ($\int_0^\infty da\, \bar{f}_{st}(a) = 1$) stationary solution $\bar{f}_{st}(a)$ is

$$\bar{f}_{st}(a) = \left(\frac{\beta-\gamma}{\gamma}\right)^{[(\bar{\alpha}-\lambda\bar{\omega})V/\gamma]+\theta} \frac{(\bar{\omega}V)^{\theta}}{\Gamma([\bar{\alpha}-\lambda\bar{\omega})V/\gamma])} e^{-a/\bar{\omega}V} a^{[(\bar{\alpha}-\lambda\bar{\omega})V/\gamma]-1}$$
$$\times M\left(\theta + \frac{(\bar{\alpha}-\lambda\bar{\omega})V}{\gamma}, \frac{(\bar{\alpha}-\lambda\bar{\omega})V}{\gamma}; y\right) \tag{4.39}$$

where $M(a, b; y)$ is the confluent hypergeometric function.[25] Substituting (4.39) in the transformation formula (2.7) it is easy to recover the stationary distribution $\bar{P}_{st}(N)$ (4.34).

The factorial moments of the distribution can be calculated either from (4.32) or (4.39)

$$\langle \overline{\Omega_m(N)} \rangle_{st} = \left.\frac{d^m \bar{F}_{st}(s)}{ds^m}\right|_{s=1} = \int_0^\infty a^m \bar{f}_{st}(a)\, da = \left(\frac{\gamma}{\beta-\gamma}\right)^m \left[\frac{(\bar{\alpha}-\lambda\bar{\omega})V}{\gamma}\right]_m$$
$$\times F\left(-m, -\theta, \frac{(\bar{\alpha}-\lambda\bar{\omega})V}{\gamma}; 1 - \frac{(\beta-\gamma)\,\bar{\omega}V}{\gamma}\right) \tag{4.40}$$

For the calculation of the first few moments it is easier to solve the equations for the moments given in general in (3.42). In this example the equation for $\langle \overline{\Omega_m(N)} \rangle_t$ is a linear equation which includes other factorial moments $\langle \overline{\Omega_l(N)} \rangle_t$ with $l \leqslant m$. In this way we can also obtain the time dependence of the factorial moments. In particular one obtains that a stationary value is reached as $t \to \infty$ only for $\beta - \gamma > 0$.

The effect of the external fluctuations of α in the statistical properties of a finite system shows up when comparing (4.40) with (4.18). In particular we find for $\langle \bar{x} \rangle_{st}$ the same value (4.19) obtained in the absence of external noise. For the relative fluctuation we have

$$\frac{\langle \overline{x^2} \rangle_{st} - \langle \bar{x} \rangle_{st}^2}{\langle \bar{x} \rangle_{st}^2} = \frac{\beta}{\bar{\alpha}V} + \frac{\lambda\bar{\omega}^2}{\bar{\alpha}^2}(\beta-\gamma) \tag{4.41}$$

In this example the relative fluctuation is just the addition of the same quantity in the absence of external noise (4.20) with the one in the thermodynamic limit (4.28). This simple result as well as the fact that $\langle \bar{x} \rangle_{st}$ is independent of the external noise parameters is in agreement with our general discussion of the equations for the moments in Section 3. It is due to the additive character of the external noise considered.

4.2.2. External Noise in the Annihilation Parameter. We now study the situation in which the control parameter in (4.15) becomes a random function $\beta \to \bar{\beta} + \xi_R(t)$. The random process $\xi_R(t)$ is again taken to be a Poisson white noise with parameter λ and an exponential distribution $\rho(\omega)$ (2.30). The physical requirement of positivity of β is guaranteed by taking $\bar{\beta} \geqslant \lambda\bar{\omega}$.

4.2.2.1. Thermodynamic Limit. The main difference with the previous case of α fluctuations is that the noise term now has a multiplicative character in the stochastic differential equation obtained when β is replaced by $\bar{\beta} + \xi_R(t)$ in (4.22). In order to discuss the stationary properties of the process $x(t)$ it is convenient to distinguish two cases: (i) $\bar{\beta} - \gamma - \lambda\bar{\omega} > 0$ and (ii) $\bar{\beta} - \gamma - \lambda\bar{\omega} < 0$. The physical difference between these two cases is quite clear. Since $\xi_R(t) \geqslant -\lambda\bar{\omega}$, in the first case the damping coefficient $\beta - \gamma$ of the deterministic equation (4.22) remains positive for all realizations of $\xi_R(t)$. The system is thus expected to reach a well-defined steady state. In the second case there are realizations of $\xi_R(t)$ for which the damping coefficient is negative. For such individual realizations (4.22) leads to a divergence of $x(t)$ as $t \to \infty$.

(i) $\bar{\beta} - \gamma - \lambda\bar{\omega} > 0$. The stationary distribution $P_{st}(x)$ is obtained from (2.31). The process has a boundary at

$$x = x_0 = \frac{\alpha}{\bar{\beta} - \gamma - \lambda\bar{\omega}} \tag{4.42}$$

The process $x(t)$ cannot reach values $x > x_0$ because $\xi_R(t)$ is bounded from below. The normalized distribution function in $(0, x_0)$ is

$$P_{st}(x) = N x^{1/\bar{\omega}} [\alpha - (\bar{\beta} - \gamma - \lambda\bar{\omega})x]^{\lambda/(\bar{\beta} - \gamma - \lambda\bar{\omega}) - 1} \tag{4.43}$$

where

$$N = \frac{(\bar{\beta} - \gamma - \lambda\bar{\omega})^{1 + 1/\bar{\omega}}}{\alpha^{\lambda/(\bar{\beta} - \gamma - \lambda\bar{\omega}) + 1/\bar{\omega}} B(\lambda/(\bar{\beta} - \gamma - \lambda\bar{\omega}), 1/\bar{\omega} + 1)} \tag{4.44}$$

and B is the β function.[25] The stationary distribution (4.43) is normalizable for all values of the parameters satisfying $\bar{\beta} - \gamma - \lambda\bar{\omega} > 0$. The formal solution of (2.31) cannot be normalized in (x_0, ∞).

All the moments of the stationary distribution (4.43) exist. They can be calculated from (4.43) or from (2.33) particularized to this example:

$$\bar{x}_{st}^m = x_0^m \frac{(1/\bar{\omega} + 1)_m}{(1 + (\bar{\beta} - \gamma)/[\bar{\omega}(\bar{\beta} - \gamma - \lambda\bar{\omega})])_m} \tag{4.45}$$

In particular for the mean value and relative fluctuation we have

$$\bar{x}_{st} = \frac{\alpha}{\beta - \gamma - \lambda\bar{\omega} + \lambda\bar{\omega}/(1+\bar{\omega})} \tag{4.46}$$

$$\frac{\overline{x_{st}^2} - \bar{x}_{st}^2}{\bar{x}_{st}^2} = \frac{\lambda\bar{\omega}^2}{(1+\bar{\omega})[(1+2\bar{\omega})(\beta-\gamma-\lambda\bar{\omega})+\lambda\bar{\omega}]} \tag{4.47}$$

The mean value (4.46) does not coincide with the deterministic stationary state $\alpha/(\beta-\gamma)$ because of the multiplicative character of the noise.

The stationary distribution (4.43) shows a transition of the same kind that we discussed in the case of fluctuations in the parameter α (Fig. 5): (4.43) implies that $P_{st}(0)=0$ for all λ and $P_{st}(x_0)=0$ for $\lambda > \beta - \gamma - \lambda\bar{\omega}$, while $P_{st}(x_0)=\infty$ for $\lambda < \beta - \gamma - \lambda\bar{\omega}$. When $P_{st}(x_0)=\infty$ there is no relative extremum of $P_{st}(x)$ but when $P_{st}(x_0)=0$ there is a maximum at

$$x_{max} = \frac{\alpha}{(\beta-\gamma-\lambda\bar{\omega})(1-\bar{\omega})+\lambda\bar{\omega}} \tag{4.48}$$

(ii) $\beta - \gamma - \lambda\bar{\omega} < 0$. When $\beta - \gamma - \lambda\bar{\omega} \to 0^+$ the boundary at x_0 goes

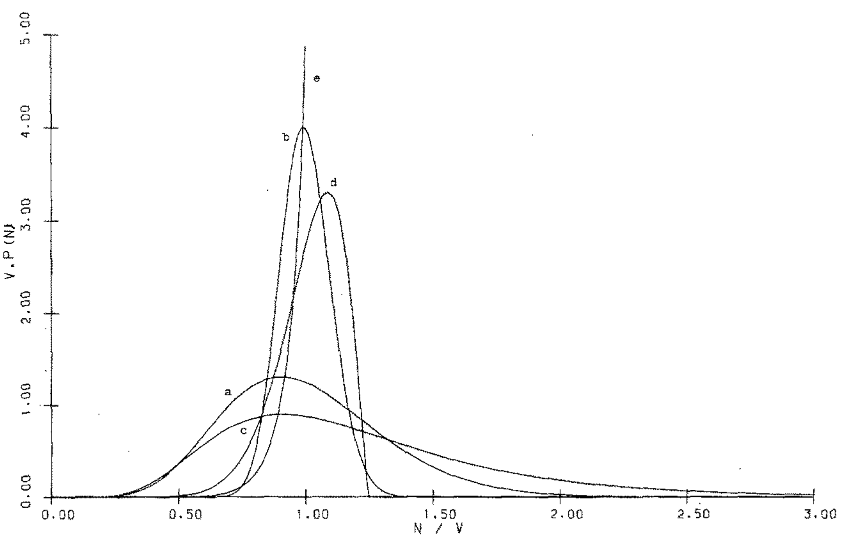

Fig. 5. Plot of different stationary probabilities comparing the effect of finite volume alone [Eq. (4.17)] with the effect of external noise in the annihilation parameter β in the thermodynamic limit [Eq. (4.43)]. Values of the parameters: $\alpha=0.1$, $\beta=1$, $\gamma=0.9$. Internal fluctuations (a) $V=100$, (b) $V=1000$. External fluctuations $\bar{\omega}=0.1$, (c) $\lambda=2$, (d) $\lambda=0.2$, (e) $\lambda=0.05$.

to infinity and for $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$ we find a stationary distribution defined in $(0, \infty)$ and normalizable for $\bar{\beta}-\gamma>0$. (Note that this condition is automatically fulfilled when $\bar{\beta}-\gamma-\lambda\bar{\omega}>0$). The stationary distribution is still given by (4.43) but with a different normalization constant

$$N'=\frac{(\gamma-\bar{\beta}+\lambda\bar{\omega})^{1/\bar{\omega}+1}}{\alpha^{\lambda/(\gamma-\bar{\beta}+\lambda\bar{\omega})-1/\bar{\omega}}B(1/\bar{\omega}+1,\ \lambda/(\gamma-\bar{\beta}+\lambda\bar{\omega})-1/\bar{\omega})} \tag{4.49}$$

In this case $P_{st}(x)$ has a maximum at $x=x_{max}$ (4.48). Therefore we find that in spite of the existence of realizations of the external noise for which $x(t)$ becomes unstable, on the average the system reaches a steady state. This steady state has an important peculiarity which can be understood as a manifestation of the destabilizing realizations in which $\bar{\beta}+\xi_R(t)-\gamma<0$: there is no value of $\lambda\bar{\omega}$ consistent with $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$ for which all the moments $\bar{x}^m_{st}$ exist. More precisely $\bar{x}^m_{st}$ diverges when $m>m_0=(\bar{\beta}-\gamma)/\bar{\omega}(\gamma-\bar{\beta}+\lambda\bar{\omega})$. When $m<m_0$, Eq. (4.45) remains valid.

4.2.2.2. Finite System. The general discussion of Section 3 applies to this case with

$$R_0(N)=\bar{\beta}N, \qquad R_1(N)=N \tag{4.50}$$

$Q_0(N)$ is given by $Q(N)$ in (4.14) and $Q_1(N)=0$. The noise parameters are $\lambda_R=\lambda$, $\bar{\omega}_R=\bar{\omega}$; $\lambda_Q=0$. The transition probabilities $\bar{W}(N, N\pm n)$ are given by (3.21)–(3.24) with these specifications and $b=1$. We now have an opposite effect to the one mentioned in the case of fluctuations in the source parameter: we now have nonvanishing transition probabilities with annihilation of n particles for all n, while no creation process with $n>1$ exists.

In this example, equation (3.33) for the generating function is

$$\frac{\partial\bar{F}(s,t)}{\partial t}=\alpha V(s-1)\,\bar{F}(s,t)+[\bar{\beta}(1-s)+\gamma(s-1)s]\,\frac{\partial\bar{F}(s,t)}{\partial s}$$
$$+\,\lambda\left[\frac{1}{1-\bar{\omega}(1-s)(\partial/\partial s)}-1-\bar{\omega}(1-s)\frac{\partial}{\partial s}\right]\bar{F}(s,t) \tag{4.51}$$

Introducing the variable

$$z=\frac{(s-1)\gamma}{\bar{\beta}-\gamma-\lambda\bar{\omega}} \tag{4.52}$$

and the parameters $t=\alpha V/\gamma$, $u=1+1/\bar{\omega}$; $v=1+(\bar{\beta}-\gamma)/[\bar{\omega}(\bar{\beta}-\gamma-\lambda\bar{\omega})]$.

The stationary generating function $\bar{F}_{st}(z)$ satisfies the hypergeometric equation

$$z(1-z)\frac{d^2\bar{F}_{st}(z)}{dz^2}+[v-(t+u+1)z]\frac{d\bar{F}_{st}(z)}{dz}-tu\bar{F}_{st}(z)=0 \quad (4.53)$$

whose general solution is[25]

$$\bar{F}_{st}(z)=AF(t,u,v;z)+Bz^{1-v}F(t-v+1,u-v+1,2-v;z) \quad (4.54)$$

The constants A and B have to be chosen in such a way that $\bar{F}_{st}(s)$ is analytic in $s=0$ and $\bar{F}_{st}(s=1)=1$. The first requirement guarantees the existence of $\bar{P}_{st}(N)$ [see (3.29)] and the second one is the condition of normalizability of $\bar{P}_{st}(N)$. As we did in the thermodynamic limit it is convenient to distinguish between the cases (i) $\bar{\beta}-\gamma-\lambda\bar{\omega}>0$ and (ii) $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$. We always take γ as a strictly positive quantity.

(i) $\bar{\beta}-\gamma-\lambda\bar{\omega}>0$. In $s=0$, $z=z_1=-\gamma/(\bar{\beta}-\gamma-\lambda\bar{\omega})<0$ and z^{1-v} diverges at $z=0$ $(s=1)$ because $1-v<0$, so that the second solution in (4.54) cannot satisfy the normalizability condition. Therefore we take $B=0$ and the normalization condition implies that $A=1$.

The probability distribution can be written as

$$\bar{P}_{st}(N)=\frac{1}{N!}\frac{d^N\bar{F}_{st}(s)}{ds^N}\bigg|_{s=0}=\frac{1}{N!}\left(\frac{\gamma}{\bar{\beta}-\gamma-\lambda\bar{\omega}}\right)^N\left(1+\frac{\gamma}{\bar{\beta}-\gamma-\lambda\bar{\omega}}\right)^{-(\alpha V/\gamma+N)}$$
$$\times\left(\frac{\alpha V}{\gamma}\right)_N\frac{(1+1/\bar{\omega})_N}{[1+(\bar{\beta}-\gamma)/\bar{\omega}(\bar{\beta}-\gamma-\lambda\bar{\omega})]_N}$$
$$\times F\left(\frac{\lambda}{\bar{\beta}-\gamma-\lambda\bar{\omega}},\frac{\alpha V}{\gamma}+N,\frac{\lambda}{\bar{\beta}-\gamma-\lambda\bar{\omega}}+\frac{1}{\bar{\omega}}+N+1;\frac{\gamma}{\bar{\beta}-\lambda\bar{\omega}}\right) \quad (4.55)$$

where $F(a,b,c;z)$ is the hypergeometric function.[25] The positivity of $\bar{P}_{st}(N)$ is guaranteed by the requirement $\bar{\beta}-\lambda\bar{\omega}\geqslant 0$. When $\bar{\beta}>\lambda\bar{\omega}$ all arguments of F in (4.55) are positive.

(ii) $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$. In this case $z_1>0$ and it is more convenient to write the general solution of (4.53) as

$$\bar{F}_{st}(z)=A'F(t,u,t+u-v+1;1-z)$$
$$+B'(1-z)^{v-t-u}F(v-u,v-t,v-t-u+1;1-z) \quad (4.56)$$

Recalling the condition $\bar{\beta}-\lambda\bar{\omega}\geqslant 0$ we observe that $z_1>1$ and therefore

$(1-z)^{v-t-u}$ has a pole at $z=1<z_1$. Therefore we set $B'=0$ and the normalization condition gives

$$A'=[F(t,u,t+u-v+1,1)]^{-1}=\frac{\Gamma(t+u-v+1)\,\Gamma(1-v)}{\Gamma(u-v+1)\,\Gamma(t-v+1)} \quad (4.57)$$

A' is only well defined when $1-v>0$ which implies $\bar{\beta}-\gamma>0$. This condition of existence of a steady state is automatically guaranteed in the case $\bar{\beta}-\gamma-\lambda\bar{\omega}>0$. It is interesting to note that this is the same condition found in the absence of external noise (4.17), in the thermodynamic limit (Section 4.2.2.1) as well as in the case of external noise in the parameter α (Section 4.2.1).

The probability distribution is given by

$$\bar{P}_{st}(N)=\frac{1}{A'N!}\frac{(\alpha V/\gamma)_N(1+1/\bar{\omega})_N}{\{\alpha V/\gamma+1/\bar{\omega}+(\bar{\beta}-\gamma)/[\bar{\omega}(\gamma-\bar{\beta}+\lambda\bar{\omega})]\}_N}\left[\frac{\gamma}{\gamma-\bar{\beta}+\lambda\bar{\omega}}\right]^N$$
$$\times F\left(\frac{\alpha V}{\gamma}-\frac{1}{\bar{\omega}}+\frac{\lambda}{\gamma-\bar{\beta}+\lambda\bar{\omega}},\ N+\frac{\alpha V}{\gamma},\right.$$
$$\left.N+\frac{\alpha V}{\gamma}+\frac{\lambda}{\gamma-\bar{\beta}+\lambda\bar{\omega}}+1;\ 1-\frac{(\gamma-\bar{\beta}+\lambda\bar{\omega})}{\gamma}\right) \quad (4.58)$$

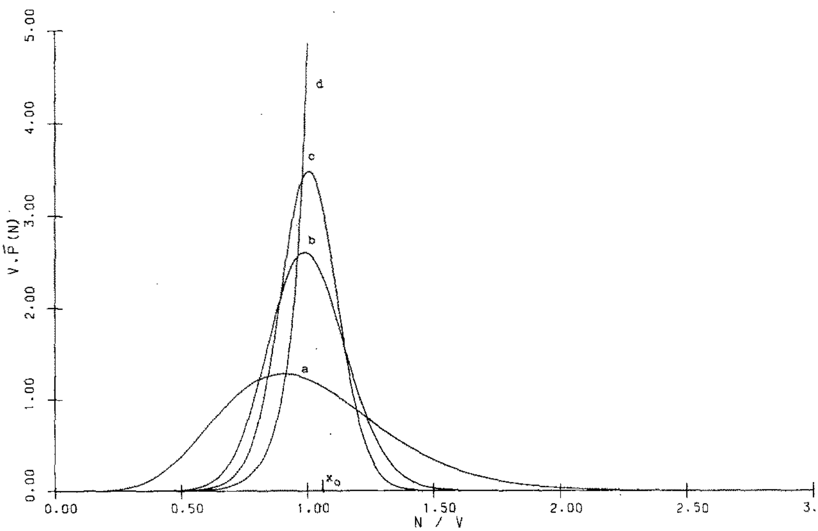

Fig. 6. Plot of the stationary probability (4.55) for different values of V and $\bar{\beta}-\gamma-\lambda\bar{\omega}>\lambda$. $\alpha=0.1$, $\bar{\beta}=1$, $\gamma=0.9$, $\bar{\omega}=0.1$, $\lambda=0.05$. (a) $V=100$, (b) $V=500$, (c) $V=1000$, (d) $V=\infty$. $x_0=1.05$.

The positivity of $\bar{P}_{st}(N)$ is guaranteed for all values of the parameters consistent with $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$ and $\bar{\beta}>\lambda\bar{\omega}$. This can be seen from the integral representation of F.[25]

In Figs. 6–8 we have plotted $\bar{P}_{st}(N)$ in different situations (see Appendix C). These figures show the combined effect of internal and external fluctuations. The same basic consequences of including internal fluctuations found in Figs. 2–4 are also seen here: the change of $\bar{P}_{st}(x)$ found at $\lambda=\bar{\beta}-\gamma-\lambda\bar{\omega}$ for $V\to\infty$ is smeared out in a finite system. A difference is now that as the noise parameter λ is increased the probability distribution moves to larger values of N/V. This is again so because $\bar{\beta}$ takes smaller values when increasing λ.

For completeness we give the equivalent solution of the problem in terms of the Poisson transform $\bar{f}_{st}(a)$. From Eq. (3.38) and with the transformation

$$\bar{f}_{st}(a)=a^{\alpha V/\gamma-1}e^{-[(\bar{\beta}-\gamma-\lambda\bar{\omega})/\gamma]a}U_1(Y_1) \tag{4.59}$$

$$Y_1=\frac{\bar{\beta}-\gamma-\lambda\bar{\omega}}{\gamma}a \tag{4.60}$$

$U_1(Y_1)$ satisfies Kummer's equation in the form

$$Y_1\frac{d^2U_1}{dY_1^2}+\left(\frac{\alpha V}{\gamma}-\frac{1}{\bar{\omega}}-Y_1\right)\frac{dU_1}{dY_1}-\frac{\lambda}{\bar{\beta}-\gamma-\lambda\bar{\omega}}U_1(Y_1)=0 \tag{4.61}$$

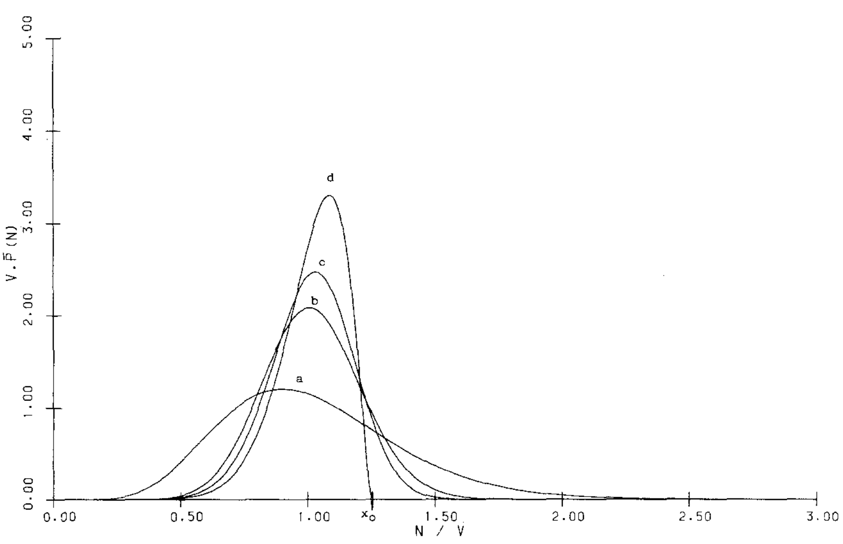

Fig. 7. Same as in Fig. 6 but with $\lambda=0.2$ $(0<\bar{\beta}-\gamma-\lambda\bar{\omega}<\lambda)$. $x_0=1.25$.

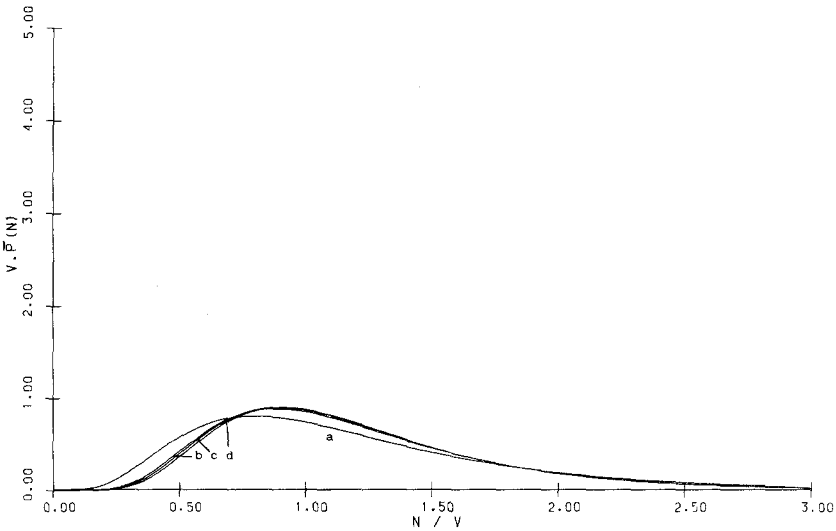

Fig. 8. Same as in Fig. 7 but with $\lambda=2$ $(\bar{\beta}-\gamma-\lambda\bar{\omega}<0)$. $x_0=0$.

Choosing the normalizable solution of this equation we find for $\bar{\beta}-\gamma-\lambda\bar{\omega}>0$

$$\bar{f}_{\rm st}(a)=\frac{\Gamma([(1+\bar{\omega})(\bar{\beta}-\gamma)-\lambda\bar{\omega}^2]/\bar{\omega}(\bar{\beta}-\gamma-\lambda\bar{\omega}))}{\Gamma(\alpha V/\gamma)\,\Gamma[(1+\bar{\omega})/\bar{\omega}]} \times\left(\frac{\bar{\beta}-\gamma-\lambda\bar{\omega}}{\gamma}\right)^{\alpha V/\gamma} a^{\alpha V/\gamma-1}e^{-[(\bar{\beta}-\gamma-\lambda\bar{\omega})/\gamma]a} \times U\left(\frac{\lambda}{\bar{\beta}-\gamma-\lambda\bar{\omega}},\frac{\alpha V}{\gamma}-\frac{1}{\bar{\omega}};\frac{\bar{\beta}-\gamma-\lambda\bar{\omega}}{\gamma}a\right) \tag{4.62}$$

where $U(a, b; c)$ is the second Kummer's solution.[25] For $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$ we find

$$\bar{f}_{\rm st}(a)=\frac{\Gamma[\alpha V/\gamma-1/\bar{\omega}+\lambda/(\gamma-\bar{\beta}+\lambda\bar{\omega})]\,\Gamma[\lambda/(\gamma-\bar{\beta}+\lambda\bar{\omega})+1]}{\Gamma(\alpha V/\gamma)\,\Gamma[\lambda/(\gamma-\bar{\beta}+\lambda\bar{\omega})-1/\bar{\omega}]\,\Gamma(1+1/\bar{\omega})} \times\left[\frac{\gamma-\bar{\beta}+\lambda\bar{\omega}}{\gamma}\right]^{\alpha V/\gamma} a^{\alpha V/\gamma-1} \times U\left(\frac{\alpha V}{\gamma}-\frac{1}{\bar{\omega}}+\frac{\lambda}{\gamma-\bar{\beta}+\lambda\bar{\omega}},\frac{\alpha V}{\gamma}-\frac{1}{\bar{\omega}};\frac{\gamma-\bar{\beta}+\lambda\bar{\omega}}{\gamma}a\right) \tag{4.63}$$

The probability distributions (4.55) and (4.58) can be recovered from (4.63), respectively.

Useful quantities to investigate the consequences of the joint description of internal and external fluctuations given here are the factorial moments $\langle\overline{\Omega_m(N)}\rangle_t$. Given an arbitrary initial condition, the time-dependent value of $\langle\overline{\Omega_m(N)}\rangle_t$ can be calculated from (3.50), which in this case is

$$\frac{d}{dt}\langle\overline{\Omega_m(N)}\rangle_t = m(\gamma-\bar{\beta})\langle\overline{\Omega_m(N)}\rangle_t + m(m-1)\,\gamma\langle\overline{\Omega_{m-1}(N)}\rangle_t + m\alpha V\langle\overline{\Omega_{m-1}(N)}\rangle_t + \frac{\lambda m^2\bar{\omega}^2}{1+m\bar{\omega}}\langle\overline{\Omega_m(N)}\rangle_t \tag{4.64}$$

Equations (4.64) are an infinite set of coupled ordinary linear differential equations which can be solved starting from $m=1$. The stationary values $\langle\overline{\Omega_m(N)}\rangle_{\text{st}}$ can also be calculated from (4.55) and (4.58) or alternatively from (4.62) and (4.63). When $\bar{\beta}-\gamma-\lambda\bar{\omega}>0$ all the factorial moments exist. They are given by

$$\langle\overline{\Omega_m(N)}\rangle_{\text{st}} = \frac{(\alpha V/\gamma)_m(1+1/\bar{\omega})_m}{[1+(\bar{\beta}-\gamma)/\bar{\omega}(\bar{\beta}-\gamma-\lambda\bar{\omega})]_m}\left(\frac{\gamma}{\bar{\beta}-\gamma-\lambda\bar{\omega}}\right)^m \tag{4.65}$$

In the case $\bar{\beta}-\gamma-\lambda\bar{\omega}<0$ the stationary factorial moments with $m \geqslant m_0=(\bar{\beta}-\gamma)/\bar{\omega}(\gamma-\bar{\beta}+\lambda\bar{\omega})$ diverge. This can also be seen from (4.64). The same result (4.65) is valid for $m<m_0$. This condition of existence of the factorial moments is the same that we found in the thermodynamic limit. Therefore, the consideration of finite size effects changes the stationary distribution and the values of the moments but does not modify the conditions for their existence. As the more important features of the probability distribution we consider the mean value and the relative fluctuation. The mean value $\langle\bar{x}\rangle_{\text{st}}$ coincides with the one find in the thermodynamic limit (4.46). It is independent of the volume of the system. This is so because $Q(N)$ is a linear function of N. For more general function $Q(N)$ we already discussed in Section 3.5 that $\langle\bar{x}\rangle_{\text{st}}$ is expected to depend on V through a "crossed-fluctuations contribution." Such crossed-fluctuations contribution is found in this example in the relative fluctuation as a consequence of the multiplicative character of the noise

$$\frac{\langle\overline{x^2}\rangle_{\text{st}}-\langle\bar{x}\rangle_{\text{st}}^2}{\langle\bar{x}\rangle_{\text{st}}^2} = \frac{\bar{\beta}}{\alpha V} + \frac{\lambda\bar{\omega}^2}{(1+\bar{\omega})[(1+2\bar{\omega})(\bar{\beta}-\gamma)-2\lambda\bar{\omega}^2]} + \frac{2\lambda^2\bar{\omega}^4-\lambda\bar{\omega}^2[(1+2\bar{\omega})\bar{\beta}-2(1+\bar{\omega})\bar{\gamma}]}{\bar{\alpha}V(1+\bar{\omega})[(1+2\bar{\omega})(\bar{\beta}-\gamma)-2\lambda\bar{\omega}^2]} \tag{4.66}$$

In (4.66) we have made explicit the three contributions which exist. The first one was already obtained in (4.20). It corresponds to the case in which there is no external noise. The second one was obtained in (4.47) and gives the relative fluctuation in the thermodynamic limit. The third term is the crossed-fluctuation contribution which comes from a coupling of internal and external fluctuations which exists in a finite system.

4.2.3. External Noise in the Creation Parameter. Finally we consider fluctuations of the parameter γ in (4.14): $\gamma \to \bar{\gamma} + \xi_Q(t)$. The process $\xi_Q(t)$ is characterized by parameters λ and $\bar{\omega}$ and has an exponential distribution (2.30). We require that $\bar{\gamma} \geqslant \lambda\bar{\omega}$. This situation differs from the case of α fluctuations in that $\xi_Q(t)$ appears now in the thermodynamic limit as a multiplicative noise. On the other hand, it is physically more closely related to α fluctuations than to β fluctuations because it produces a random creation of particles.

4.2.3.1. Thermodynamic Limit. The stochastic differential equation is now given by (4.22) with γ replaced by $\bar{\gamma} + \xi_Q(t)$. The main difference with the case of β fluctuations analyzed in Section 4.2.2 is that now, for all values of λ and $\bar{\omega}$ there are realizations of the noise in which $\beta - \gamma$ becomes negative, leading to divergent trajectories of $x(t)$. Nevertheless, a stationary distribution exists which can be obtained from (2.31). The process has now a boundary at

$$x_0 = \frac{\alpha}{\beta - \bar{\gamma} + \lambda\bar{\omega}} \tag{4.67}$$

The noise $\xi_Q(t)$ appears now in the stochastic differential equation with a different sign that for β fluctuations. As a consequence the process $x(t)$ cannot reach now values $x < x_0$ similarly to what happened for α fluctuations. Thus the stationary distribution is defined in the interval (x_0, ∞):

$$P_{st}(x) = N x^{-1/\bar{\omega}} [(\beta - \bar{\gamma} + \lambda\bar{\omega}) x - \alpha]^{\lambda/(\beta - \bar{\gamma} + \lambda\bar{\omega}) - 1} \tag{4.68}$$

This distribution can be normalized if $\beta - \bar{\gamma} > 0$ and the normalization constant is

$$N = \frac{x_0^{1/\bar{\omega} - 1}}{B(1/\bar{\omega} - \lambda x_0/\alpha, \lambda x_0/\alpha)\, \alpha^{\lambda x_0/\alpha - 1}} \tag{4.69}$$

By the same reasons that in the case of β fluctuations with $\beta - \gamma - \lambda\bar{\omega} < 0$, there are no values of the parameters for which all the moments $\overline{x_{st}^m}$ exist. For $m > (\beta - \bar{\gamma})/\bar{\omega}(\beta - \bar{\gamma} + \lambda\bar{\omega}) = m_0$, $\overline{x_{st}^m}$ diverges. This is easily seen from (4.68) or (2.33). For $m < m_0$ we find

$$\overline{x_{st}^m} = x_0^m \frac{B(1/\bar{\omega} - \lambda x_0/\alpha - m, \lambda x_0/\alpha)}{B(1/\bar{\omega} - \lambda x_0/\alpha, \lambda x_0/\alpha)}$$

$$= \frac{x_0^m \Gamma(1/\bar{\omega} - \lambda x_0/\alpha - m)\, \Gamma(1/\bar{\omega})}{\Gamma(1/\bar{\omega} - m)\, \Gamma(1/\bar{\omega} - \lambda x_0/\alpha)} \tag{4.70}$$

and in particular

$$\bar{x}_{st} = \frac{\alpha}{\beta - \bar{\gamma} + \lambda\bar{\omega} - \lambda\bar{\omega}/(1 - \bar{\omega})} \tag{4.71}$$

$$\frac{\overline{x_{st}^2} - \bar{x}_{st}^2}{\bar{x}_{st}^2} = \frac{\lambda\bar{\omega}^2}{(1 - \bar{\omega})[(\beta - \bar{\gamma} + \lambda\bar{\omega})(1 - 2\bar{\omega}) - \lambda\bar{\omega}]} \tag{4.72}$$

The stationary distribution (4.68) shows the same sort of transition discussed for α and β fluctuations now at $\lambda = \beta - \bar{\gamma} + \lambda\bar{\omega}$ (Fig. 9): $P_{st}(x_0) = \infty$ if $\lambda < \beta - \bar{\gamma} + \lambda\bar{\omega}$, while $P_{st}(x_0) = 0$ if $\lambda > \beta - \bar{\gamma} + \lambda\bar{\omega}$. In the first

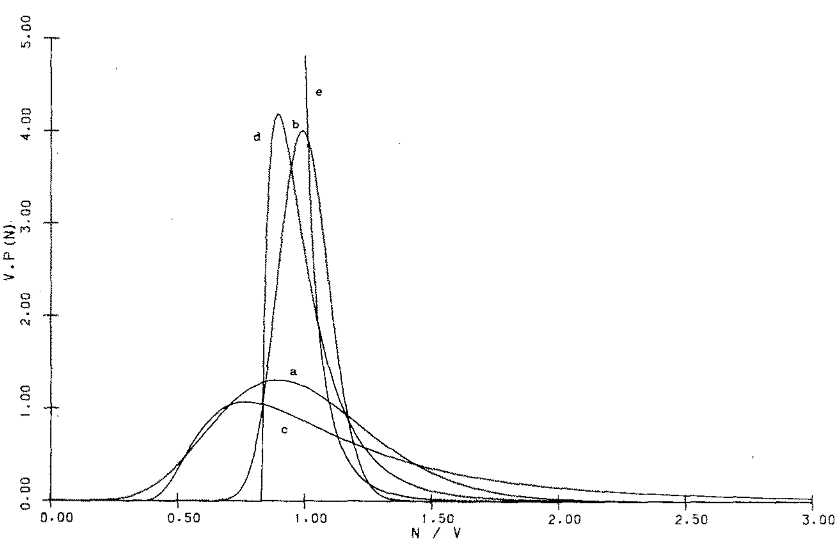

Fig. 9. Plot of different stationary probabilities comparing the effect of finite volume alone [Eq. (4.17)] with the effect of external noise in the creation parameter γ in the thermodynamic limit [Eq. (4.68)]. Same values of parameters than in Fig. 5.

situation there is no relative extremum of $P_{st}(x)$ and in the second one there is a maximum at

$$x_{max} = \frac{\alpha}{(\beta - \bar{\gamma} + \lambda\bar{\omega})(1+\bar{\omega}) - \lambda\bar{\omega}} \tag{4.73}$$

In spite of the different origin of the fluctuations, the similarity of the distributions shown in Figs. 9 and 1 is quite remarkable.

We finally note that there are important differences in the stationary distributions obtained in the case of β and γ fluctuations. Even the interval of definition is different. In this respect we note that if we were to use Gaussian white noise instead of Poisson white noise, we would find no difference between β and γ fluctuations. In the two cases there is no value of the Gaussian white-noise parameter for which all the stationary moments exist. The positivity requirement of β and γ cannot be fulfilled with a Gaussian white noise. As a consequence there are always realizations in which the process is unstable and it is not possible to distinguish between fluctuations of the creation parameter γ and fluctuations of the annihilation parameter β.

4.2.3.2. Finite System. In the notation of Section 3 we now have

$$Q_0(N) = \alpha V + \bar{\gamma} N, \qquad Q_1(N) = N \tag{4.74}$$

$R_0(N)$ is given by $R(N)$ in (4.15). The transition probabilities $W(N, N \pm n)$ are given by (3.21)–(3.24) with $\lambda_Q = \lambda$; $\lambda_R = 0$; $\omega_Q = \omega$ and $a = 1$. In the same way that in the case of fluctuations of the source parameter there exist nonvanishing transition probabilities with creation of n particles for all n, while the annihilation process is only possible for $n = 1$.

The equation for the averaged generating function (3.30) is in this case

$$\frac{\partial \bar{F}(s,t)}{\partial t} = \alpha V(s-1)\,\bar{F}(s,t) + [\beta(1-s) + \bar{\gamma} s(s-1)]\,\frac{\partial \bar{F}(s,t)}{\partial s} + \lambda\left[\frac{1}{1-\bar{\omega}(s-1)\,s(\partial/\partial s)} - 1 - \bar{\omega}(s-1)\,s(\partial/\partial s)\right]\bar{F}(s,t) \tag{4.75}$$

We have not found a steady state analytical solution of (4.75). Nevertheless, substituting (3.29) in (4.75) leads to a recursion relation which permits a numerical calculation of $\bar{P}_{st}(N)$ (see Appendix C). Results are shown in Figs. 10–12. As in previous cases, internal fluctuations smear out the change of shape of $P_{st}(x)$ found in the thermodynamic limit at $\lambda = \beta - \bar{\gamma} + \lambda\bar{\omega}$. For small values of V internal fluctuations dominate the stationary distribution which is essentially the same for any of the three

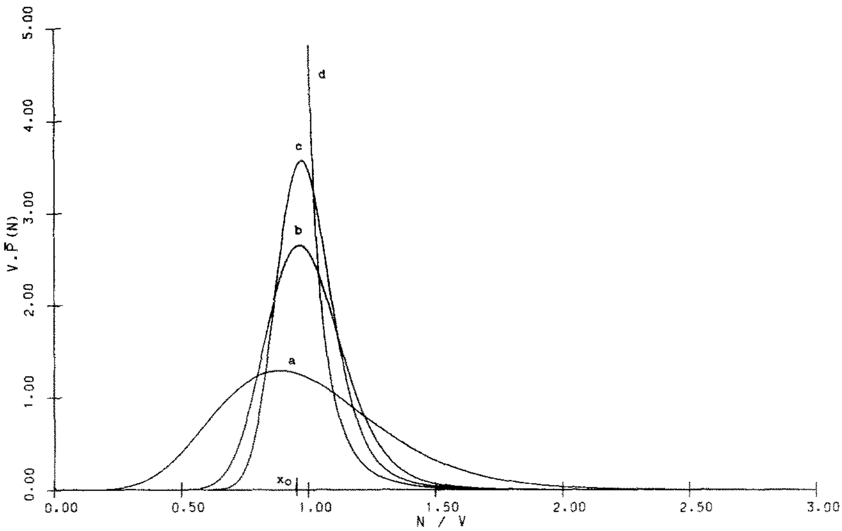

Fig. 10. Stationary probability (C2) corresponding to fluctuations of the parameter γ for different values of V and $\beta-\bar{\gamma}+\lambda\bar{\omega}>\lambda$. Same values of parameters than in Fig. 6. $x_0=0.95$.

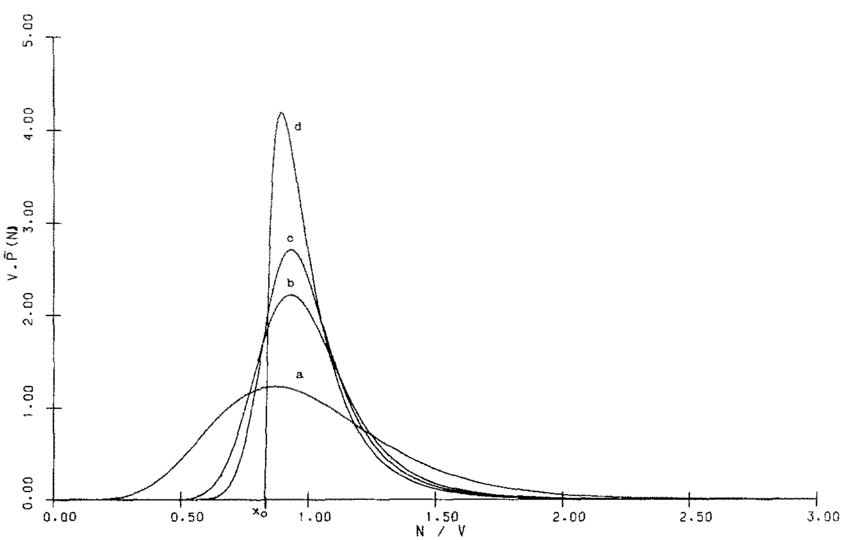

Fig. 11. Same as in Fig. 10 but with $\lambda=0.2$ $(\beta-\bar{\gamma}+\lambda\bar{\omega}<\lambda)$. $x_0=0.83$.

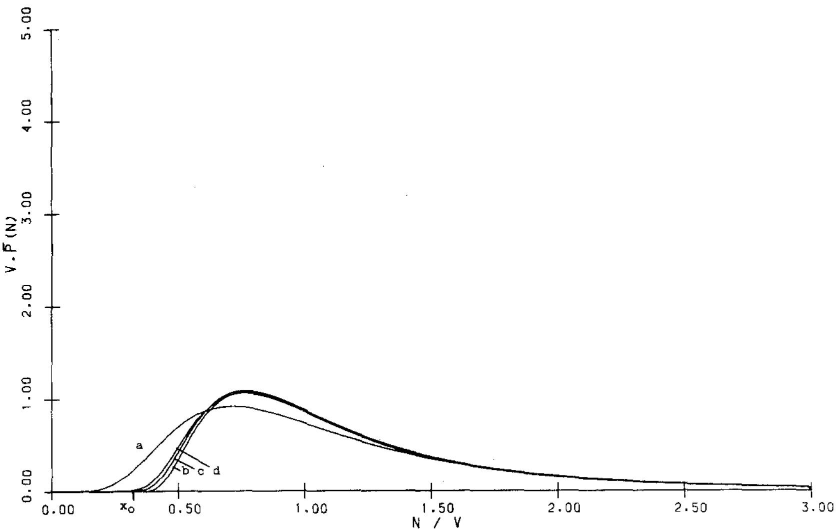

Fig. 12. Same as in Fig. 10 but with $\lambda = 2$ $(\beta - \bar{\gamma} + \lambda\bar{\omega} < \lambda)$. $x_0 = 0.33$.

cases of external noise that we have considered. On the other hand, it is interesting to note that for finite V, fluctuation in the source parameter α or in the creation parameter γ lead to very similar stationary distributions which are notably different from the ones corresponding to fluctuations in the annihilation parameter (compare Figs. 10, 11 with Figs. 2, 3 and 6, 7). This is most remarkable given the fact that, as already mentioned, a standard treatment in the thermodynamic limit with Gaussian white noise does not distinguish β fluctuations from γ fluctuations.

The factorial moments $\langle \overline{\Omega_m(N)} \rangle$ can be analyzed from Eq. (3.50). We first note that not all the moments exist in the stationary state. From (3.50) it is immediate to see that the factorial moments with $m > m_0 = (\beta - \bar{\gamma})/\bar{\omega}(\beta - \bar{\gamma} + \lambda\bar{\omega})$ diverge in the steady state. For $m < m_0$ a recursion relation can be given for the stationary factorial moments. To this end, it is convenient to start one step before (3.50): taking the average over ω with an exponential distribution in (3.48) and substituting in (3.41) we have

$$\begin{aligned} \frac{d}{dt} \langle \overline{\Omega_m(N)} \rangle_t = {} & m(\bar{\gamma} - \beta) \langle \overline{\Omega_m(N)} \rangle_t + m(m-1)\,\bar{\gamma} \langle \overline{\Omega_{m-1}(N)} \rangle_t \\ & + m\alpha V \langle \overline{\Omega_{m-1}(N)} \rangle_t \\ & + \lambda \left[\frac{1}{1 - \bar{\omega} m (1 + D^- m)} - \bar{\omega} m (1 + D^- m) - 1 \right] \langle \overline{\Omega_m(N)} \rangle_t \end{aligned} \tag{4.76}$$

Solving in the steady state we find the following recursion relation:

$$[(1-\bar{\omega}m)(\beta-\bar{\gamma})-\lambda\bar{\omega}^2 m]\langle\overline{\Omega_m(N)}\rangle_{st}+\{-\bar{\omega}(m-1)^2(\beta-\bar{\gamma})-(1-\bar{\omega}m)$$
$$\times[(m-1)\bar{\gamma}+\alpha V]-\lambda\bar{\omega}^2(m-1)(2m-1)\}\langle\overline{\Omega_{m-1}(N)}\rangle_{st}$$
$$+\bar{\omega}(m-1)^2[(m-2)\bar{\gamma}+\alpha V-\lambda\bar{\omega}(m-2)]\langle\overline{\Omega_{m-2}(N)}\rangle_{st}=0 \qquad (4.77)$$

This relation permits a recursive calculation of all the moments of $\bar{P}_{st}(N)$. In particular we recover (4.71) for the mean value. This coincidence is for the same reasons discussed in the case of β fluctuations. The relative fluctuations are given by

$$\frac{\langle\overline{x^2}\rangle_{st}-\langle\bar{x}\rangle_{st}^2}{\langle\bar{x}\rangle_{st}^2}=\frac{\beta}{\alpha V}+\frac{\lambda\bar{\omega}^2}{(1-\bar{\omega})[(-2\bar{\omega}+1)(\beta-\bar{\gamma})-2\lambda\bar{\omega}^2]}$$
$$+\frac{\lambda\bar{\omega}^2[(1-\bar{\omega})(2\beta-\bar{\gamma})-\lambda\bar{\omega}^2]}{\alpha V(1-\bar{\omega})^2[(1-2\bar{\omega})(\beta-\bar{\gamma})-2\lambda\bar{\omega}^2]} \qquad (4.78)$$

where a crossed-fluctuation term of the same nature than in (4.66) has been made explicit.

ACKNOWLEDGMENT

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APPENDIX A

A proper definition of white Poisson noise $z^w(t)$ is given by a limit in which the duration of the pulses of a generalized Poisson process $z(t)$ (shot noise) goes to zero.[12] This last process is defined by

$$z(t)=\sum_{i=1}^{n(t)}\omega_i h(t-t_i) \qquad (A1)$$

where $n(t)$ is a Poisson counting process with probability

$$P(n(t)=n)=e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad (A2)$$

and the t_i are uniformly distributed in the interval $(0, t)$. The function $h(t-t_i)$ is a pulse attached to the time t_i such that $h(t-t_i)=0$ for $t<t_i$. The pulses are weighted by ω_i which are random independent variables

with a probability distribution $\rho(\omega)$. The characteristic functional of $z(t)$ is[14,26]

$$\Phi_t[v] = \exp\left[\lambda \int_0^t dt_1 \left\{\exp\left[i\omega \int_{t_1}^t ds\, h(s-t_1)\, v(s)\right]\right\}_{\text{av}} - 1\right] \tag{A3}$$

In the limit in which the pulses $h(t-t_i)$ become δ functions (A3) becomes the generating functional (2.24) which defines white Poisson noise.[10,14]

From (A3) and (2.24) the cumulant generating functionals $\psi_t[v]$ and $\psi_t^w[v]$ are, respectively,

$$\psi_t[v] = \lambda \sum_{n=1}^{\infty} \frac{i^n}{n!} \{\omega^n\}_{\text{av}} \int_0^t dt_1 \int_{t_1}^t ds_1 \cdots \int_{t_1}^t ds_n\, h(s_1 - t_1) \cdots h(s_n - t_1) \times v(s_1) \cdots v(s_n) \tag{A4}$$

$$\psi_t^w[v] = \lambda \sum_{n=0}^{\infty} \frac{i^n}{n!} \{\omega^n\}_{\text{av}} \int_0^t ds\, v^n(s) \tag{A5}$$

Defining the cumulants of the process $z(t)$ in the standard way by

$$k_n(t_1,..., t_n) = i^{-n} \frac{\delta^n \psi_t[v]}{\delta v(t_1) \cdots \delta v(t_n)}\bigg|_{v=0} \tag{A6}$$

and the same formula for $z^w(t)$ with ψ_t replaced by $\varphi_t^w[v]$ we find

$$k_n(t_1,..., t_n) = \lambda \{\omega^n\}_{\text{av}} \int_0^t ds\, h(t_1 - s) \cdots h(t_n - s) \tag{A7}$$

$$k_1^w(t_1) = \lambda\{\omega\}_{\text{av}}, \qquad k_n^w(t_1,..., t_n) = \lambda\{\omega^n\}_{\text{av}}\, \delta(t_1 - t_2)\, \delta(t_1 - t_3) \cdots \delta(t_1 - t_n) \tag{A8}$$

An equivalent definition of the white Poisson noise $z^w(t)$ is given[14] by the stochastic differential equation

$$\dot{y}(t) = z^w(t) \tag{A9}$$

where the probability density $P(y, t)$ of the process $y(t)$ satisfies the master equation

$$\frac{\partial P(y, t)}{\partial t} = \lambda \int P(y-\omega, t)\, \rho(\omega)\, d\omega - \lambda P(y, t) \tag{A10}$$

The average of $z(t)$ with a functional $u[z]$ can be calculated from the general formula[10]

$$\langle z(t)\,u[z]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n\, k_{n+1}(t, t_1,..., t_n) \times \left\langle \frac{\delta^n u[z]}{\delta z(t_1) \cdots \delta z(t_n)} \right\rangle \tag{A11}$$

Substituting (A7) and taking a rectangular pulse $h(t)$

$$h(t) = \begin{cases} 0, & t<0,\ \varepsilon<t \\ A, & 0<t<\varepsilon \end{cases} \tag{A12}$$

we find

$$\langle z(t)\,u[z]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda\{\omega^{n+1}\}_{\text{av}} \int_{t-\varepsilon}^{t} ds \int_s^t dt_1 \cdots \int_s^t dt_n \times A^{n+1} \left\langle \frac{\delta^n u[z]}{\delta z(t_1) \cdots \delta z(t_n)} \right\rangle \tag{A13}$$

With the change of variables $\tau_i = (t - t_i)/s$, $\tau = (t-s)/\varepsilon$, (A13) is written

$$\langle z(t)\,u[z]\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda\{\omega^{n+1}\}_{\text{av}} \int_0^1 d\tau \int_0^\tau d\tau_1 \cdots \int_0^\tau d\tau_n (A\varepsilon)^{n+1} \times \left\langle \frac{\delta^n u[z]}{\delta z(t-\varepsilon\tau_1) \cdots \delta z(t-\varepsilon z_n)} \right\rangle \tag{A14}$$

The white noise limit in which $z(t) \to z^w(t)$ is given by $\varepsilon \to 0$, $A \to \infty$ with $\varepsilon A = 1$. In this limit

$$\langle z^w(t)\,u[z]\rangle = \lambda\{\omega\}_{\text{av}} + \sum_{n=1}^{\infty} \frac{1}{n!} \lambda\{\omega^{n+1}\}_{\text{av}} \int_0^1 d\tau\, \tau^n \left\langle \frac{\delta^n u[z]}{\delta z^n(t)} \right\rangle \tag{A15}$$

which reproduces (2.25). This formula can also be directly obtained from the general result of Hanggi[10] which is valid for white noise processes:

$$\langle z^w(t)\,u[z]\rangle = \left\langle \sum_t \left[i \frac{\delta}{\delta z} \right] u[z] \right\rangle \tag{A16}$$

where

$$\sum_t \left[i \frac{\delta}{dt} \right] = \frac{1}{i\theta(t)} \frac{\partial}{\partial t} \psi_t^w[v] \tag{A17}$$

and $\psi_t^w[v]$ is given by (A5). We finally remark[10] that a direct substitution of (A8) in (A11) for the white noise process does not reproduce the correct formula (2.25).

The relation of white Poisson noise with the dichotomic Markov process and the Gaussian white noise has been studied by Van den Broeck.[16]

APPENDIX B

In this appendix we derive the equalities used in (3.17), (3.18) and Eq. (3.57). Both operator relations can be obtained from the following operator identity proven below:

$$\exp\{A\} \exp\{B\} = \exp\left[(e^{\omega} - 1) \frac{A+B}{\omega} \right] \tag{B1}$$

where A and B are two operators whose commutator is

$$[A, B] = \omega(A+B) \tag{B2}$$

(3.17) follows from (B1) with $A = \omega_Q aN$, $B = \omega_Q a(E^- - 1)N$ and $\omega = \omega_Q a$. Equation (3.18) follows from (B1) with $A = \omega_R bN$, $B = \omega_R b(E^+ - 1)N$ and $\omega = -\omega_R b$. Finally, (3.57) is obtained from (B1) with $A = -a\omega_Q m$, $B = a\omega_R m(1 + D^- m)$ and $\omega = -a\omega_Q$.

In order to get (B1) we use the Campbell–Hausdorff formula[27]

$$\exp\{A\} \exp\{B\} = \exp\{\eta(A, B)\} \tag{B3}$$

with $\eta(A, B)$ given by the following expression:

$$\eta(A, B) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \sum_{\substack{p_i + q_i \geq 1 \\ i = 1,\ldots,m}} \frac{[\overbrace{A \cdots A}^{p_1}\overbrace{B \cdots B}^{q_1} \cdots \overbrace{B \cdots B}^{q_m}]}{[\sum_{j=1}^{m} (p_j + q_j)] \prod_{k=1}^{m} (p_k!\, q_k!)} \tag{B4}$$

where

$$[CDE \cdots \mathscr{T}] = [\cdots[[C, D], E] \cdots \mathscr{T}] \tag{B5}$$

By using (B2) we obtain

$$[A] = A, \qquad [B] = B, \qquad [A\overbrace{B \cdots B}^{q_1}] = \omega^{q_1}(A+B) \quad (q_1 > 0) \tag{B6}$$

and

$$[A\overbrace{B\cdots B}^{q_1}\cdots\overbrace{B\cdots B}^{q_m}]$$
$$=\omega^{\sum_2^m (p_i+q_i)}\omega^{q_1}(-1)^{\sum_2^m p_i}(A+B)[1-\delta_{q_1,0}(1-\delta_{p_2,0})], \qquad (p_1=1) \tag{B7}$$

$$[B\overbrace{A\cdots A}^{p_2}\cdots\overbrace{B\cdots B}^{q_m}]$$
$$=\omega^{\sum_2^m (p_i+q_i)}(-1)^{\sum_2^m p_i}(A+B)(1-\delta_{p_2,0}), \qquad (p_1=0,\, q_1=1) \tag{B8}$$

where $m\geqslant 2$, $p_i+q_i\geqslant 1$, and $\delta_{a,b}$ is the Kronecker delta. Then we have

$$\eta(A, B)=\sum_{n=0}^{\infty}\frac{\omega^n}{(n+1)!}(A+B)+\sum_{m\geqslant 2}\frac{(-1)^{m-1}}{m}$$
$$\times\left[\sum_{\substack{p_i+q_i\geqslant 1\\ i=2,\ldots,m\\ q_1\geqslant 0}}\frac{\omega^{\sum_2^m (p_i+q_i)}\omega^{q_1}(-1)^{\sum_2^m p_i}[1-\delta_{q_1,0}(1-\delta_{p_2,0})]}{[\sum_2^m (p_i+q_i)+1+q_1]\prod_{k=2}^m (p_k!\,q_k!)\,q_1!}\right]$$
$$\times (A+B)+\sum_{m\geqslant 2}\frac{(-1)^{m-1}}{m}$$
$$\times\left[\sum_{\substack{p_i+q_i\geqslant 1\\ i=2,\ldots,m}}\frac{\omega^{\sum_2^m (p_i+q_i)}(-1)^{\sum_2^m p_i}(1-\delta_{p_2,0})}{[\sum_2^m (p_i+q_i)+0+1]\prod_{k=2}^m (p_k!\,q_k!)}\right](A+B) \tag{B9}$$

If we write the last term on the right-hand side of (B9) in the following form:

$$\sum_{m\geqslant 2}\frac{(-1)^{m-1}}{m}\left[\sum_{\substack{q_1\geqslant 0, p_i+q_i\geqslant 1\\ i=2,\ldots,m}}\frac{\omega^{\sum_2^m (p_i+q_i)}\omega^{q_1}(-1)^{\sum_2^m p_i}(1-\delta_{p_2,0})\,\delta_{q_1,0}}{[\sum_2^m (p_i+q_i)+q_1+1]\prod_{k=2}^m (p_k!\,q_k!)\,q_1!}\right]$$
$$\times (A+B) \tag{B10}$$

we have

$$\eta(A, B)=(e^{\omega}-1)\frac{(A+B)}{\omega}+\sum_{m\geqslant 2}\frac{(-1)^{m-1}}{m}$$
$$\times\left[\sum_{\substack{q_1\geqslant 0, p_i+q_i\geqslant 1\\ i=2,\ldots,m}}\frac{\omega^{\sum_2^m (p_i+q_i)}\omega^{q_1}(-1)^{\sum_2^m p_i}}{[\sum_2^m (p_i+q_i)+q_1+1]\prod_{k=2}^m (p_k!\,q_k!)\,q_1!}\right]$$
$$\times (A+B) \tag{B11}$$

Now, if we show that the second term in (B11) is zero we obtain (B1). We can write this term in powers of ω in the following form:

$$\sum_{m\geqslant 2}\frac{(-1)^{m-1}}{m}\sum_{n=m}^{\infty}\frac{\omega^{n-1}}{n}\sum_{\substack{n_i\geqslant 1\\ \sum_{i=1}^{m}n_i=n}}\times\left\{\left[\sum_{q_1=0}^{n_1-1}\frac{1}{q_1!}\right]\prod_{i=2}^{m}\left[\sum_{\substack{p_i,q_i\\ p_i+q_i=n_i}}\frac{(-1)^{p_i}}{p_i!\,q_i!}\right]\right\}(A+B) \tag{B12}$$

which vanishes because

$$\sum_{\substack{p_i,q_i\\ p_i+q_i=n_i\geqslant 1}}\frac{(-1)^{p_i}}{p_i!\,q_i!}=\frac{(1-1)^{n_i}}{n_i!}=0 \tag{B13}$$

APPENDIX C

In this appendix we explain the numerical procedure followed to calculate the probabilities shown in Figs. 1–12. For the purpose of computation it is more suitable the use of recurrence solutions than the calculation of the analytic formulas (4.34), (4.55), (4.58).

For a noisy α parameter (Figs. 2–4) the following recurrence has been employed:

$$\bar{P}(N)=\frac{[\beta V\bar{\omega}+\gamma(1+V\bar{\omega})](N-1)+\bar{\alpha}V(1+V\bar{\omega})-\lambda V^2\bar{\omega}^2}{N\beta(1+V\bar{\omega})}\bar{P}(N-1)$$
$$+\frac{(\lambda V^2\bar{\omega}^2-\gamma V\bar{\omega}(N-2)-\bar{\alpha}V^2\bar{\omega})}{}\bar{P}(N-1),\qquad N\geqslant 2 \tag{C1}$$

$$\bar{P}(1)=\frac{\bar{\alpha}V(1+V\bar{\omega})-\lambda V^2\bar{\omega}^2}{\beta(1+V\bar{\omega})}\bar{P}(0)$$

This recurrence can be obtained taking derivatives in (4.31) and substituting $\partial^N\bar{F}(s)/\partial s^N=N!\,\bar{P}(N)$. The initial value $\bar{P}(0)$ is calculated by normalization.

In a similar way we have computed the probability for the case noise in the parameter γ (Figs. 10–12). From Eq. (4.75) we have obtained the following recurrence relation:

$$\bar{P}(N)=\frac{\left[\begin{array}{l}\beta\bar{\omega}(N-1)^2+\bar{\gamma}(N-1)[1+\bar{\omega}(N-1)] \\ \qquad +\alpha V[1+\bar{\omega}(N-1)]-\lambda\omega^2(N-1)^2\end{array}\right]}{N\beta[1+\bar{\omega}(N-1)]}\bar{P}(N-1)$$

$$+\frac{\bar{\omega}(N-1)[-\bar{\gamma}(N-2)-\alpha V+\lambda\bar{\omega}(N-2)]}{N\beta[1+\bar{\omega}(N-1)]}\bar{P}(N-2), \quad N\geqslant 2 \quad \text{(C2)}$$

$$\bar{P}(1)=\frac{\alpha V}{\beta}\bar{P}(0)$$

When the noise is in the parameter β (Figs. 6–8) we have from (4.51) the recurrence relation:

$$\bar{P}(N)=\frac{\bar{\beta}[1+\bar{\omega}(N-1)]+\gamma\bar{\omega}(N-1)+\alpha V\bar{\omega}-\lambda\bar{\omega}^2(N-1)}{N(N-1)(\bar{\beta}-\lambda\bar{\omega})\bar{\omega}}\bar{P}(N-1)$$

$$-\frac{\{\gamma(N-2)[1+\bar{\omega}(N-1)]+\alpha V[1+\bar{\omega}(N-1)]\}}{N(N-1)(\bar{\beta}-\lambda\bar{\omega})\bar{\omega}}\bar{P}(N-2), \quad N\geqslant 2 \quad \text{(C3)}$$

In this case there are two unspecified quantities, $\bar{P}(1)$ and $\bar{P}(0)$ and we have only one condition of normalization. The difference with the above cases lies in the fact that the noise in the absorption parameter β produces destruction operators in the equivalent master equation. For the calculation of probabilities we use now a method based on a continued fraction expansion as follows. We introduce the neighboring ratio as[28]

$$\xi_N=\frac{\bar{P}(N)}{\bar{P}(N-1)} \quad \text{(C4)}$$

Operating in (C3) we obtain the continued fraction relation:

$$\xi_N=\frac{A(N)}{B(N)-\xi_{N+1}} \quad \text{(C5)}$$

where

$$A(N)=\frac{\gamma(N+1)(1+\bar{\omega}N)+\alpha V(1+\bar{\omega}N)}{(N+1)\,N(\bar{\beta}-\lambda\bar{\omega})\bar{\omega}} \quad \text{(C6)}$$

$$B(N)=\frac{\bar{\beta}(1+\bar{\omega}N)+\gamma\bar{\omega}N+\alpha V\bar{\omega}-\lambda\bar{\omega}^2N}{N(\bar{\beta}-\lambda\bar{\omega})\bar{\omega}} \quad \text{(C7)}$$

The quantities ξ_N are numerically calculated with the use of standard approximations.[29] Now, the probabilities are given by

$$\bar{P}(N) = \bar{P}(0) \prod_{i=1}^{N} \xi_i$$

where as in the other cases the zero probability $\bar{P}(0)$ is obtained by normalization.

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